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CONCRETE CATEGORIES AND INFINITARY LANGUAGES

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Introduction

By definition, a concrete category is a category of sets which are endowed with an unspecified structure. There have been some attempts to make this structure specific. For example, Blanchard [6] used Bourbaki-type structures and Kučera and Pultr [17] have determined the structure by a functor Set→Set. Our aim is to consider concrete categories as categories of models of first-order theories. However, for these theories to be a syntactic counterpart of concrete categories, they must exceed the usual ones in the following three points: nonlogical symbols are of arbitrary arities, there may be a proper class of them and infinitary logical symbols are admitted. This language might be called 'an unrestricted $L_{\infty,\infty}$'. Its strength is illustrated by the fact that we may imagine any concrete category as a category consisting of models of this language.

Of course, in this generality one cannot expect very model-theoretic results. Our approach is more an approach of 'a working mathematician', i.e. to see the relation between syntactic properties of theories and semantical properties of their categories of models. This approach has a practical aspect, i.e. to know what may be said about current models from life (e.g. when there exist limits, colimits, free objects or when models form a cartesian closed category, a symmetric monoidal closed one etc.). There is also a theoretical aspect, i.e. to characterize theories providing a given categorical property or on the contrary to characterize concrete categories of models of theories of a prescribed kind. The pattern of these 'categorical preservation theorems' is the Beck–Linton theorem (see MacLane [20, p. 147]) which covers the equational case. A characteristic feature of these preservation theorems is the fact that they work outside any similarity type. The classical preservation theorem corresponding to the above mentioned Beck–Linton one is Birkhoff's characterization of varieties of algebras of a given type.

The generality of the syntax used means that the categories of models arising need not be legitimate. In Section 2 we establish a smallness property of a theory which ensures the legitimacy of its category of models. Moreover, these theories correspond to strongly fibre-small concrete categories in the sense of Adámek, Herrlich and Strecker [1]. The first section contains preliminaries. All undefined concepts from category theory may be found in MacLane [20] and concerning infinitary languages Dickmann [9] is recommended.

The third section deals with the passage from semantics to syntax. The main idea is to assign to a concrete category $\mathscr{A}$ a suitable canonical language which contains a card($|A|$)-ary relation symbol R_A for each object A of $\mathscr{A}$. The last two sections contain the categorical preservation theorems relating initially and semi-initially complete categories to certain Horn theories. Finally, the fourth section is concerned with the question, following naturally from preceding considerations, of what can be said about theories with isomorphic concrete categories of models.

We emphasize that no attention is paid to languages with some smallness condition – neither to the arities of nonlogical symbols, nor to their number, nor to the strength of the used logical connectives and quantifiers. Further we restrict ourselves to sets as a base category. It seems that all can be done over a general base category $\mathscr{T}$ by using languages adapted to $\mathscr{T}$. Here, arities are objects of $\mathscr{T}$; the equational languages of this kind have been touched by Linton [9] (see [29] for a further development).

The present paper continues and refines the previous investigations of the author (see [26, 27, 28]). Nearly all results here included were reported at the Summer School on Algebra and Ordered Sets, Vsetín 1978.

1. Preliminaries

We will work in Gödel–Bernays set theory with the axiom of choice for classes. A *category* $\mathscr{A}$ is given by a class of objects and sets of morphisms $\mathscr{A}(A,B)$ for any objects A, B of $\mathscr{A}$. The case that there is more than one class of objects, which exceeds GB, will be used only informally and we will then speak about a *metacategory*.

By a *concrete category* we will understand a category $\mathscr{A}$ equipped with a faithful functor $|\ |:\mathscr{A}\to\mathrm{Set}$ into the category of sets such that the following two conditions are satisfied:

(1) If $A\in\mathscr{A}$, X is a set and $f:|A|\to X$ a bijection, then there is a $B\in\mathscr{A}$ and an isomorphism $g:A\to B$ such that $|B|=X$ and $|g|=f$.

(2) If $A,B\in\mathscr{A}$ and $f:A\to B$ is an isomorphism such that $|A|=|B|$ and $|f|=1_{|A|}$, then $A=B$ and $f=1_A$.

Here $|A|$ denotes the underlying set of an object $A\in\mathscr{A}$ and $|g|$ the underlying mapping of a morphism g. In what follows, we will identify a morphism $g:A\to B$ with its underlying mapping $|g|$. We say in this case that a mapping $g:|A|\to|B|$ is a morphism $A\to B$. A concrete category $(\mathscr{A},|\ |)$ itself will be briefly denoted by $\mathscr{A}$. If $|\ |$ has a left adjoint then we say that $\mathscr{A}$ *has free objects*. Two concrete categories $\mathscr{A}$, $\mathscr{B}$ are *isomorphic* if there is an isomorphism $F:\mathscr{A}\to\mathscr{B}$ which commutes with the underlying set functors (i.e. $|\ |\cdot F=|\ |$). The last condition means that F is a *concrete functor* (or a functor over Set). In what follows we will work only with concrete functors between concrete categories. Henceforth the adjective concrete will be omitted. It is well known that conditions (1) and (2) remove the distinction between equivalent and isomorphic concrete categories as the following assertion shows.

1.1. Lemma. *Any full functor $F: \mathcal{A} \to \mathcal{B}$ between concrete categories is a full embedding. If, in addition, for any $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ such that FA and B are isomorphic, then F is an isomorphism.*

Condition (2) ensures that any object $A \in \mathcal{A}$ is isomorphic to an object of $\mathcal{A}$ of which the underlying set is a cardinal. The class of these objects will be denoted by $\mathcal{A}_*$, i.e. $\mathcal{A}_* = \{A \in \mathcal{A} \mid |A| \in \mathrm{Card}\}$. Similarly, $\mathcal{A}_n = \{A \in \mathcal{A} \mid \|A\| = n\}$. Here Card denotes the class of all cardinals. Ord will similarly denote the class of all ordinals. The symbol n^+ denotes the successor of a cardinal n. The cardinality of a set X is denoted by $|X|$, hence $\|A\|$ is the cardinality of the underlying set of an object $A \in \mathcal{A}$.

By a *type* τ we will understand a quadruple $\tau = \langle \mathrm{Rel}(\tau), \mathrm{Fnt}(\tau), a_{\mathrm{Rel}(\tau)}, a_{\mathrm{Fnt}(\tau)} \rangle$ where $\mathrm{Rel}(\tau)$ and $\mathrm{Fnt}(\tau)$ are classes and $a_{\mathrm{Rel}(\tau)}: \mathrm{Rel}(\tau) \to \mathrm{Card}$ and $a_{\mathrm{Fnt}(\tau)}: \mathrm{Fnt}(\tau) \to \mathrm{Card}$ are mappings. Elements of $\mathrm{Rel}(\tau)$ are called *relation symbols* of type τ, elements of $\mathrm{Fnt}(\tau)$ *function symbols* of type τ and mappings $a_{\mathrm{Rel}(\tau)}$ and $a_{\mathrm{Fnt}(\tau)}$ assign to each relation and function symbol its *arity*. $\mathrm{Rel}_n(\tau)$ or $\mathrm{Fnt}_n(\tau)$ will denote the class of all n-ary relation or function symbols respectively. 0-ary function symbols are called *constants* and 0-ary relation symbols are also called *propositional constants*.

We emphasize that a proper class of (infinitary) relation and function symbols is permitted. A type will be called *small* if $\mathrm{Rel}(\tau) \cup \mathrm{Fnt}(\tau)$ is a set and *large* in the other case. We will sometimes pass from a given type τ to another type σ which has all the symbols of τ plus some additional symbols. In such cases we will use the notation $\tau \subset \sigma$ and we say that σ is an *expansion* of τ (or that τ is a *reduction* of σ). In the special case where σ is obtained from τ by adding a set of constants, σ is said to be a *simple expansion* of τ.

We will need a proper class V of *variables*. If n is a set, $x \in V^n$ is a mapping $n \to V$, and $i \in n$, then $x(i)$ will be denoted by x_i. If $n > 0$ is a cardinal, then we get an n-tuple x of variables and its ith component x_i. The case $n = \phi$ gives the empty string of variables which enables us to treat propositional constants in the same manner as the other relation symbols. Having $x \in V^n$ and $a \in n^k$ then we may form the composition $x \cdot a \in V^k$.

Let τ be a type. We are going to describe the language $L_{\kappa,\lambda}(\tau)$ of type τ. Indices κ, λ may be infinite cardinals or a symbol $\infty \notin \mathrm{Card}$. We assume that $n < \infty$ for any cardinal n.

Terms of $L_{\kappa,\lambda}(\tau)$ are defined as follows:

(3) Any variable is a term.

(4) If $f \in \mathrm{Fnt}_n(\tau)$ and $t = (t_i)_{i \in n}$ is an n-tuple of terms, then $f(t)$ is a term.

(5) All terms are defined by applications of (3) and (4).

We emphasize that following (4) any constant is a term.

Atomic formulas of $L_{\kappa,\lambda}(\tau)$ are strings of the form given below:

(6) $t_1 = t_2$ is an atomic formula where t_1 and t_2 are terms.

(7) If $R \in \mathrm{Rel}_n(\tau)$ and $t = (t_i)_{i \in n}$ is an n-tuple of terms then $R(t)$ is an atomic formula.

Finally, the *formulas* of $L_{\kappa,\lambda}(\tau)$ are defined as follows:

(8) An atomic formula is a formula.

(9) If φ is a formula then $\neg\varphi$ is a formula. If n is a set such that $0 < |n| < \kappa$ and φ_i is a formula for each $i \in n$ then $\bigwedge_{i \in n} \varphi_i$ is a formula.

(10) If n is a set such that $|n| < \lambda$, $x \in V^n$, and φ is a formula, then $(\exists x)\varphi$ is a formula.

(11) All formulas are defined by applications of (8)–(10).

Note that the symbols $\bigvee_{i \in n}$, $\Rightarrow$, $\Leftrightarrow$ and $(\forall x)$, where $x \in V^n$, are introduced as usual.

A *sentence* is a formula without free variables. A class of sentences of $L_{\kappa,\lambda}(\tau)$ is called a *theory* of $L_{\kappa,\lambda}(\tau)$.

Let us emphasize that we use $\varphi(x, y, \ldots)$, where $x \in V^n$, $y \in V^m, \ldots$, to denote a formula whose free variables are among x_i, $i \in n$, y_j, $j \in m, \ldots$. The language $L_{\omega,\omega}$ is the usual first-order language. All languages $L_{\kappa,\lambda}$ are instances of $L_{\infty,\infty}$ which admits conjunctions and disjunctions of arbitrary nonempty set of formulas and quantifications over arbitrary strings of variables. In what follows, all syntactical concepts will be related to $L_{\infty,\infty}$; only a type will be denoted explicitly. E.g. a theory of type τ means a theory of $L_{\infty,\infty}(\tau)$. We will occasionally need a larger language $L_{\infty^-,\infty}$ admitting class-indexed conjunctions and disjunctions, as well.

A formula is called *quantifier-free* if no quantifiers are built in it. A formula $(\forall x)\varphi(x)$ or $(\forall x)(\exists y)\varphi(x,y)$, where φ is quantifier-free, is called *universal* or *universal-existential* respectively. A *universal* (*universal-existential*) *theory* consists of universal (universal-existential) sentences. An *existential-positive formula* contains only $\wedge$, $\vee$ and $\exists$. A $\wedge$*-formula* means a conjunction of atomic formulas. A formula $(\exists x)\varphi(x)$, where φ is a $\wedge$-formula, is called an $\exists\wedge$*-formula*. Finally, $(\exists! x)\varphi(x)$ is the abbreviation of

$$(\exists x)\varphi(x) \wedge (\forall y, z)(\varphi(y) \wedge \varphi(z) \Rightarrow y = z).$$

A *model* of type τ is a triple $\mathfrak{A} = \langle A, S, F\rangle$ where A is a set, S is a mapping defined on $\mathrm{Rel}(\tau)$ which assigns to each $R \in \mathrm{Rel}_n(\tau)$ an n-ary relation $S(R) \subseteq A^n$ on A, and F is a mapping defined on $\mathrm{Fnt}(\tau)$ and assigning to each $f \in \mathrm{Fnt}_n(\tau)$ a function $F(f): A^n \to A$.

In what follows the relation $S(R)$ will be denoted by $R_{\mathfrak{A}}$ and the function $F(f)$ by $f_{\mathfrak{A}}$. The underlying set A will be denoted by $|\mathfrak{A}|$. We emphasize that empty models (i.e. $A = \emptyset$) are admitted. Because of the presence of propostional constants there may be more than one empty model. Indeed, a propositional constant is interpreted as a subset $R_{\mathfrak{A}} \subseteq A^0$ and there are two possibilities: either $R_{\mathfrak{A}} = \emptyset$ or $R_{\mathfrak{A}} = A^0$.

If $\tau \subseteq \sigma$ then a model $\mathfrak{A}$ of type σ determines a *reduct* of $\mathfrak{A}$ to τ. A *homomorphism* $h: \mathfrak{A} \to \mathfrak{B}$ of models of type τ is a map $h: |\mathfrak{A}| \to |\mathfrak{B}|$ such that $h \cdot a \in R_{\mathfrak{B}}$ for any $R \in \mathrm{Rel}(\tau)$ and $a \in R_{\mathfrak{A}}$ and $h \cdot f_{\mathfrak{A}}(a) = f_{\mathfrak{B}}(h^n(a))$ for any $f \in \mathrm{Fnt}_n(\tau)$ and $a \in |\mathfrak{A}|^n$. A model $\mathfrak{A}$ is called a *submodel* of $\mathfrak{B}$ if $|\mathfrak{A}| \subseteq |\mathfrak{B}|$, $R_{\mathfrak{A}}$ is the restriction to $\mathfrak{A}$ of $R_{\mathfrak{B}}$ for any $R \in \mathrm{Rel}(\tau)$ and $f_{\mathfrak{A}}$ is the restriction to $\mathfrak{A}$ of $f_{\mathfrak{B}}$ for any $f \in \mathrm{Fnt}(\tau)$.

The notion of *satisfaction* is defined as usual. If $\mathfrak{A}$ is a τ-model, $\varphi(x)$ a formula of type τ, $x \in V^n$ and $a \in |\mathfrak{A}|^n$, then $\mathfrak{A} \models \varphi[a]$ will denote that a satisfies φ in $\mathfrak{A}$. Let T be a theory of type τ. A τ-model $\mathfrak{A}$ is a *model* of T if all sentences of T are satisfied in $\mathfrak{A}$.

Let $\mathfrak{A}$ be a model of type τ and $x \in V^{|\mathfrak{A}|}$. The class $\Delta_{\mathfrak{A}} = \{\varphi(x) \mid \varphi$ is an atomic formula or a negation of an atomic formula and $\mathfrak{A} \models \varphi[1_{|\mathfrak{A}|}]\}$ is called the *diagram* of $\mathfrak{A}$. If we take into account atomic formulas φ only then we get the *positive diagram* $\Delta^+_{\mathfrak{A}}$ of $\mathfrak{A}$. Clearly $f: |\mathfrak{A}| \to |\mathfrak{B}|$ is a homomorphism of models iff $\mathfrak{B} \models \varphi[f]$ for any $\varphi(x) \in \Delta^+_{\mathfrak{A}}$. On the other hand, the *elementary diagram* $\Delta^e_{\mathfrak{A}}$ appears if we consider all formulas φ.

2. Categories of models

Models of type τ and homomorphisms between them form a concrete metacategory $\mathrm{Mod}(\tau)$. Conditions (1) and (2) are evidently satisfied. The legitimacy of $\mathrm{Mod}(\tau)$ is treated in the next proposition. Recall that a concrete category $\mathscr{A}$ is *fibre-small* if $\{A \in \mathscr{A} \mid |A| = X\}$ is a set for any set X.

2.1. Proposition. $\mathrm{Mod}(\tau)$ *is a category iff* τ *is small. In this case* $\mathrm{Mod}(\tau)$ *is moreover fibre-small.*

Proof. If τ is small then $\mathrm{Mod}(\tau)$ is evidently a fibre-small concrete category. Let τ be large and assume that $\mathrm{Rel}(\tau)$ is a proper class (if $\mathrm{Fnt}(\tau)$ is a proper class, the argument is analogous). Then either $C = \mathrm{Rel}_n(\tau)$ is a proper class for some n or there is a proper class C of cardinals n such that $\mathrm{Rel}_n(\tau) \neq \emptyset$. In any case, the proper metaclass of all subclasses of C can be injectively mapped into the metaclass of all τ-models on a set $X \neq \emptyset$ because there are two different k-ary relations on X for any cardinal k. Hence $\mathrm{Mod}(\tau)$ is not a category.

Denote by $\mathrm{Mod}(T)$ the concrete metacategory of all models of a theory T (together with homomorphisms, again). Of course, there may now be a theory T of a large type such that $\mathrm{Mod}(T)$ is a category. We give such an example with the additional property that $\mathrm{Mod}(T)$ is not fibre-small.

2.2. Example. Let τ consist of unary relation symbols R_i where i runs over all ordinals. Let T consist of sentences

$$(\forall x)(R_i(x) \Rightarrow R_j(x)) \quad \text{for } i \leq j,\ i, j \in \mathrm{Ord},$$

$$(\exists x)(\forall y)(y = x) \qquad \text{where } x, y \in V.$$

Then T only has one-element models $\mathfrak{A}$, $\mathfrak{A}_n$ where $n \in \mathrm{Ord}$ given by: $(R_i)_{\mathfrak{A}} = \emptyset$ for any $i \in \mathrm{Ord}$ and $(R_i)_{\mathfrak{A}_n} = \emptyset$ iff $n > i$ for any $i, n \in \mathrm{Ord}$.

A theory T is called *equational* if its type contains function symbols only and T consists of universally quantified atomic formulas. An example of an equational theory of $L_{\omega,\omega}$ with the above property is given in Reiterman [24]. It contains unary function symbols f_i, $i \in \mathrm{Ord}$, and sentences $(\forall x) f_i(f_j(x)) = f_{\max(i,j)}(x)$.

The problem of the characterization of theories T such that $\mathrm{Mod}(T)$ is a category is open. We will however find a natural kind of theories with well-behaved $\mathrm{Mod}(T)$.

Let T be a theory of type τ. A sentence φ of type τ is a *consequence* of T if $\mathfrak{A} \models \varphi$ for any T-model $\mathfrak{A}$. Notationally, $T \models \varphi$. We say that two formulas $\varphi(x)$ and $\psi(x)$ of type τ are *T-equivalent* if $T \models (\forall x)(\varphi(x) \Leftrightarrow \psi(x))$. We write $\varphi \sim_T \psi$. Having a class S of sentences of type τ and $S_0 \subseteq S$ such that for any $\varphi \in S$ there is a unique $\psi \in S_0$ T-equivalent to φ, then we say that S_0 is a *T-representative class* of S.

2.3. Definition. A theory T of type τ will be called *normal* if for each cardinal n and each $x \in V^n$ the class of all atomic formulas $\varphi(x)$ has a T-representative set.

Any theory of a small type is normal. A theory is normal iff for each cardinal n and each $x \in V^n$ there is a T-representative set of quantifier-free formulas $\varphi(x)$.

If $\mathfrak{A}$ is a model of a normal theory T then its diagram $\Delta_{\mathfrak{A}}$ has a T-representative set $\Delta'_{\mathfrak{A}}$. Hence we have the formula $\delta_{\mathfrak{A}}(x) = \bigwedge_{\varphi \in \Delta'_{\mathfrak{A}}} \varphi(x)$. Similarly, the positive diagram $\Delta^+_{\mathfrak{A}}$ yields the formula $\delta^+_{\mathfrak{A}}(x)$. If $\mathfrak{B}$ is now a T-model and $f: |\mathfrak{A}| \to |\mathfrak{B}|$ a mapping then $\mathfrak{B} \models \delta^+_{\mathfrak{A}}[f]$ iff $\mathfrak{B} \models \varphi[f]$ for any $\varphi \in \Delta^+_{\mathfrak{A}}$. Hence

$$f: \mathfrak{A} \to \mathfrak{B} \text{ is a homomorphism iff } \mathfrak{B} \models \delta^+_{\mathfrak{A}}[f]. \tag{2.1}$$

2.4. Proposition. *Let T be a normal theory of type τ. Then τ-submodels of T-models form a fibre-small concrete category.*

Proof. Assign the formula $\delta^+_{\mathfrak{A}}$ to each submodel $\mathfrak{A}$ of a T-model. Since $a \in |\mathfrak{B}|^n$ satisfies a universal formula α in a submodel iff it satisfies α in the whole model, (2.1) holds even for any submodel $\mathfrak{B}$ of a T-model.

Consider two submodels $\mathfrak{A}$, $\mathfrak{B}$ of T-models such that $|\mathfrak{A}| = |\mathfrak{B}|$ and $\delta^+_{\mathfrak{A}}$, $\delta^+_{\mathfrak{B}}$ are T-equivalent. Since $\mathfrak{A} \models \delta^+_{\mathfrak{A}}[1_{|\mathfrak{A}|}]$, the same argument concerning universal formulas as above shows that $\mathfrak{B} \models \delta^+_{\mathfrak{A}}[1_{|\mathfrak{A}|}]$ and thus $1_{|\mathfrak{A}|}: \mathfrak{A} \to \mathfrak{B}$ is a homomorphism. Similarly for $1_{|\mathfrak{A}|}: \mathfrak{B} \to \mathfrak{A}$. Hence $\mathfrak{A} = \mathfrak{B}$.

Since for any $x \in V^n$ there is a T-representative set of quantifier-free formulas, the proof is complete.

A normal theory has, by this result, a fibre-small $\mathrm{Mod}(T)$. However, more can be said, and to do it we need strong fibre-smallness introduced by Adámek, Herrlich and Strecker [1].

Let $\mathcal{A}$ be a concrete category and X a set. A *structured map into* X is a pair (A, f) consisting of $A \in \mathcal{A}$ and of a mapping $f: |A| \to X$. It will be briefly denoted by $f: |A| \to X$. A class S of structured maps into X is called a *sink* to X. Dually we define *structured maps from* X as pairs (f, A) such that $A \in \mathscr{A}$ and $f: X \to |A|$. A *source* from X is a class of structured maps from X.

Let S be a sink to X. Let $\bar{S}$ be the source from X consisting of all structured maps $h: X \to |C|$ such that $h \cdot f: A \to C$ is a morphism for any $f: |A| \to X$ from S. Two sinks S_1 and S_2 to X will be called *equivalent* if $\bar{S}_1 = \bar{S}_2$. Notationally, $S_1 \sim S_2$. For $f: |A| \to X$ and $g: |B| \to X$ we write briefly $f \sim g$ instead of $\{f\} \sim \{g\}$. A concrete category $\mathscr{A}$ is called *strongly fibre-small* if for any sink S there is a sink $S' \subseteq S$, equivalent to S, which is a set. Evidently, $\mathscr{A}$ is strongly fibre-small iff for any set X there is only a set (up to $\sim$) of structured maps into X or equivalently there is only a set, up to $\sim$, of sinks to X. This concept is further self-dual, i.e. $\mathscr{A}$ is strongly fibre-small iff there is always only a set of structured maps from X. Any strongly fibre-small concrete category is fibre-small (see [1]).

2.5. Proposition. *Let T be a theory of type τ such that submodels of T-models form a fibre small concrete category. Then* Mod(T) *is strongly fibre-small.*

Proof. Assume that there is a set X and a proper class of mutually nonequivalent structured maps $f_i: |\mathfrak{A}_i| \to X$ into X where $\mathfrak{A}_i$ are T-models. We may suppose that $f_i(|\mathfrak{A}_i|) = Y$ for any $i \in I$. For any $i, j \in I$, $i \neq j$, there is $h_{ij}: X \to |\mathfrak{B}_{ij}|$, where $\mathfrak{B}_{ij} \in \mathrm{Mod}(T)$, such that exactly one of mappings $h_{ij} \cdot f_i: \mathfrak{A}_i \to \mathfrak{B}_{ij}$, $h_{ij} \cdot f_j: \mathfrak{A}_j \to \mathfrak{B}_{ij}$ is a homomorphism. Hence $h_{ij}(Y)$ underlies a submodel $\mathfrak{D}_{ij}$ of $\mathfrak{B}_{ij}$. Since there is only a set of such $\mathfrak{D}_{ij}$, we may assume that $\mathfrak{D}_{ij} = \mathfrak{D}$ for any $i, j \in I$. Let H_i consist of all mappings $g: Y \to |\mathfrak{D}|$ such that $g \cdot f_i: \mathfrak{A}_i \to \mathfrak{D}$ is a homomorphism. There are $i \neq j$ such that $H_i = H_j$. But h_{ij} belongs to the symmetric difference $H_i \div H_j$, which is a contradiction.

2.6. Corollary. Mod(T) *is a strongly fibre-small concrete category for any normal theory T.*

Proof. This follows by Propositions 2.4 and 2.5.

2.7. Corollary. *Let T be a universal theory and* Mod(T) *be fibre-small. Then* Mod(T) *is strongly fibre-small.*

Proof. If T is universal then any submodel of a T-model is a T-model. Hence the result follows by Proposition 2.5.

The last coincidence of the two concepts of fibre-smallness follows also from a result of Kučera and Pultr [17] (see [2, Th. 2.11] as well). It may happen that T is not normal but Mod(T) is strongly fibre-small. An example of an equational theory of $L_{\omega,\omega}$ with this property appears in Reiterman [25, Ex. 3.1]. It contains a 0-ary function symbol 0, unary function symbols f_i indexed by all ordinals, a binary function symbol $+$ and sentences $(\forall x)(x + x = 0)$, $(\forall x)(0 + x = 0)$ and $(\forall x)(f_i(x) +$

$f_j(x) = f_{\max(i,j)}$) for any ordinals i, j. It is proved in [25] that Mod(T) is fibre-small and that $y = f_i(x)$, $i \in$ Ord, is a class of mutually non-T-equivalent formulas. It remains to apply Corollary 2.7.

Nevertheless, we will see in the next section that any strongly fibre-small concrete category is isomorphic to the category of models of a normal theory.

3. Representation of concrete categories

Let $\mathscr{A}$ be a concrete category. If n is a cardinal then $|\ |^n$ will denote the functor Set$(n, |\text{-}|) : \mathscr{A} \to$ Set. Our aim is to represent $\mathscr{A}$ as a category of models of some type τ. Relation and function symbols of τ cannot be arbitrarily chosen because any $R \in \mathrm{Rel}_n(\tau)$ determines a subfunctor $\bar{R}$ of $|\ |^n$ ($\bar{R}(\mathfrak{A}) = \{a : n \to |\mathfrak{A}| / a \in R_{\mathfrak{A}}\}$ for any $\mathfrak{A} \in \mathrm{Mod}(\tau)$) and any $f \in \mathrm{Fnt}_n(\tau)$ yields analogously a natural transformation $\bar{f} : |\ |^n \to |\ |$. This observation, commonly exploited in the equational case (see e.g. Linton [18]) and indicated in the general case by Manes [21], leads us to the following definition.

Let $\mathscr{A}$ be a concrete category and n a cardinal. Subfunctors of $|\ |^n$ will be called *n-ary relation symbols interpretable in $\mathscr{A}$* and natural transformations $|\ |^n \to |\ |$ *n-ary function symbols interpretable in $\mathscr{A}$*. Let $\varrho_{\mathscr{A}}$ be the metaclass of all relation and function symbols interpretable in $\mathscr{A}$. We emphasize that $\varrho_{\mathscr{A}}$ need not be a type because it may be a proper metaclass.

As usual, we may treat interpretable function symbols as a special case of relation ones. Instead of subfunctors of $|\ |^n$ we may work with corresponding sources from n. More generally, sources from n could be called n-ary formulas interpretable in $\mathscr{A}$. The treatment of cones as formulas is developped by Andréka, Németi and Sain [3].

Let $\tau \subseteq \varrho_{\mathscr{A}}$ be a type. There is a functor $G_\tau : \mathscr{A} \to \mathrm{Mod}(\tau)$ such that if $A \in \mathscr{A}$ then $|G_\tau(A)| = |A|$, $R_{G_\tau(A)} = R(A)$ for any subfunctor $R \in \mathrm{Rel}_n(\tau)$ and $f_{G_\tau(A)}$ is the component f_A of a natural transformation $f \in \mathrm{Fnt}_n(\tau)$.

In what follows, we will only work with interpretable relation symbols R_C where $C \in \mathscr{A}_*$. (Recall that $\mathscr{A}_* = \{C \in \mathscr{A} \mid |C| \in \mathrm{Card}\}$.) They are given by the prescription $R_C(A) = \{|f| \mid f : C \to A\}$ for any $A \in \mathscr{A}$.

Let $\mathscr{C} \subseteq \mathscr{A}_*$. The type $\tau_{\mathscr{C}}$ consisting of relation symbols R_C where $C \in \mathscr{C}$ will be called a *canonical type* of $\mathscr{A}$ with respect to $\mathscr{C}$. The functor $G_{\tau_{\mathscr{C}}}$ will be briefly denoted by $G_{\mathscr{C}}$. We emphasize that $(R_C)_{G_{\mathscr{C}}(A)} = \{|f| \mid f : C \to A\}$. To advance further some notions will be useful.

A concrete category $\mathscr{A}$ is *initially complete* if for each source $S = (f_i : X \to |A_i|)_{i \in I}$ there exists an object $A \in \mathscr{A}$ (called the *initial lift* of S) such that:

(a) $|A| = X$ and $f_i : A \to A_i$ is a morphism for any $i \in I$,

(b) if for a given $h : |B| \to X$ every $f_i \cdot h : B \to A$ is a morphism, then $h : B \to A$ must also be a morphism.

A class $\mathscr{C}$ of objects (or a subcategory $\mathscr{C}$) of $\mathscr{A}$ is called *initially dense* if each object of $\mathscr{A}$ is an initial lift of some source with codomains in $\mathscr{C}$. Dual notions:

finally complete, final lift and *finally dense*. All these concepts can be found e.g. in [1]. It is well known that initial and final completeness coincide.

$\mathscr{C} \subseteq \mathscr{A}$ will be called *strongly finally dense* if for any cardinal n there is a subset $\mathscr{C}_n \subseteq \mathscr{C}$ such that each object $A \in \mathscr{A}$ with $\|A\| = n$ is a final lift of some sink with domains in $\mathscr{C}_n$. Of course, strong final density implies final density.

3.1. Lemma. *If $\mathscr{A}$ is strongly fibre-small and $\mathscr{C} \subseteq \mathscr{A}$ finally dense then $\mathscr{C}$ is strongly finally dense.*

Proof. Let n be a cardinal and consider the sink S consisting of all mappings $f: |C| \to n$ with $C \in \mathscr{C}$. Since $\mathscr{A}$ is strongly fibre-small, there is a set $S' \subseteq S$ equivalent to S. It suffices to collect into $\mathscr{C}_n$ the domains of structured maps from S'.

3.2. Lemma. *A concrete category $\mathscr{A}$ has a strongly finally dense class of objects iff it is fibre-small.*

Proof. If $\mathscr{A}$ is fibre-small then $\mathscr{C} = \mathscr{A}_*$ is strongly finally dense. Indeed, it suffices to put $\mathscr{C}_n = \mathscr{A}_n$.

Let $\mathscr{C} \subseteq \mathscr{A}$ be strongly finally dense. Let n be a cardinal. If $A, B \in \mathscr{A}$, $|A| = |B| = n$, are not isomorphic, then $\{f: C \to A \mid C \in \mathscr{C}_n\}$ and $\{f: C \to B \mid C \in \mathscr{C}_n\}$ are distinct subsets of a set of all structured maps (C, f) into n with $C \in \mathscr{C}_n$. Hence $\mathscr{A}$ is fibre-small.

3.3. Proposition. *Let $\mathscr{C}$ be a finally dense class of objects of a concrete category $\mathscr{A}$. Then the functor $G_{\mathscr{C}}: \mathscr{A} \to \mathrm{Mod}(\tau_{\mathscr{C}})$ is a full embedding.*

Proof. Consider a homomorphism $h: G_{\mathscr{C}}(A) \to G_{\mathscr{C}}(B)$ and a morphism $f: C \to A$ where $C \in \mathscr{C}$. Then $f \in (R_C)_{G_{\mathscr{C}}(A)}$ and thus $h \cdot f \in (R_C)_{G_{\mathscr{C}}(B)}$. Hence $h \cdot f: C \to B$ is a morphism and since $\mathscr{C}$ is finally dense, $h: A \to B$ is a morphism. It remains to use Lemma 1.1.

In what follows, we will only work with finally dense $\mathscr{C}$ and thus we can always identify A with $G_{\mathscr{C}}(A)$. Therefore, $\mathscr{A}$ will be considered as a full subcategory of $\mathrm{Mod}(\tau_{\mathscr{C}})$. For $C \in \mathscr{C}$ and $A \in \mathscr{A}$ the symbol $(R_C)_A$ will hence mean the interpretation of the relation symbol R_C in the model $A = G_{\mathscr{C}}(A)$.

3.4. Corollary. *Any concrete category is isomorphic to a full subcategory of some* $\mathrm{Mod}(\tau)$.

Proof. The class of all objects of $\mathscr{A}$ is finally dense in $\mathscr{A}$.

It was our admission of 0-ary relation symbols and empty models which enabled us to treat the empty set in the same way as others. Indeed, without empty models we should have to restrict Corollary 3.4 to concrete categories $\mathscr{A}$ such that $|A| \neq \emptyset$ for any $A \in \mathscr{A}$. Without 0-ary relation symbols, $\mathscr{A}$ would have only one object on the empty set which would moreover be initial.

For $\mathscr{C} \subsetneq \mathscr{A}_*$, let $T_{\mathscr{C}}$ be the theory of type $\tau_{\mathscr{C}}$ consisting of all sentences which hold in all models $G_{\mathscr{C}}(A)$ where $A \in \mathscr{A}$. $T_{\mathscr{C}}$ will be called the *canonical theory* of $\mathscr{A}$ with respect to $\mathscr{C}$. $G_{\mathscr{C}}$ is clearly a functor $\mathscr{A} \to \mathrm{Mod}(T_{\mathscr{C}})$.

Any sentence

$$(\forall x)(R_C(x) \Rightarrow R_D(x \cdot f)), \tag{3.1}$$

where $C, D \in \mathscr{C}$, $f: D \to C$ is a morphism, and $x \in V^{|C|}$ belongs to $T_{\mathscr{C}}$. A model $\mathfrak{A}$ satisfying (3.1) determines a subfunctor $\tilde{\mathfrak{A}}$ of the contravariant functor $\mathrm{Set}(|\text{-}|, \mathfrak{A}): \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ by means of $\tilde{\mathfrak{A}}(C) = (R_C)_{\mathfrak{A}}$. Therefore $\tau_{\mathscr{A}}$-models satisfying (3.1) are precisely the sieves of Antoine and Wyler (see [4, 5, 33]). Proposition 3.3 generalizes Antoine's corresponding result which contains, in addition, the following important fact.

3.5. Lemma. *Let $\mathscr{C} \subseteq \mathscr{A}_*$ be finally dense, $\mathfrak{A}$ a $\tau_{\mathscr{C}}$-model satisfying* (3.1) *$C \in \mathscr{C}$ and $a \in |\mathfrak{A}|^{|C|}$. Then $a \in (R_C)_{\mathfrak{A}}$ iff $a: C \to \mathfrak{A}$ is a homomorphism.*

Proof. If a is a homomorphism then $a \in (R_C)_{\mathfrak{A}}$ because $1_C \in (R_C)_C$. Let $a \in (R_C)_{\mathfrak{A}}$ and $f: D \to C$ be a homomorphism. By (3.1) $a \cdot f \in (R_D)_{\mathfrak{A}}$ and thus $a: C \to \mathfrak{A}$ is a homomorphism.

3.6. Proposition. *Let $\mathscr{C} \subseteq \mathscr{A}_*$ be strongly finally dense. Then $G_{\mathscr{C}}: \mathscr{A} \to \mathrm{Mod}(T_{\mathscr{C}})$ is an isomorphism.*

Proof. Let $\mathfrak{B}$ be a $T_{\mathscr{C}}$-model such that $|\mathfrak{B}| = n$ is a cardinal. Following Proposition 3.3 and Lemma 1.1 it suffices to find $B \in \mathscr{A}$ such that $B \cong \mathfrak{B}$.

Denote by $\theta_k(x)$ the formula

$$\bigwedge_{\substack{i,j \in k \\ i \neq j}} (x_i \neq x_j) \wedge (\forall y) \bigvee_{i \in k} (y = x_i)$$

where $x \in V^k$ and $0 < k \in \mathrm{Card}$. Evidently, $\mathfrak{A} \vDash (\exists x)\theta_k(x)$ iff $\|\mathfrak{A}\| = k$ for any $\mathfrak{A} \in \mathrm{Mod}(T_{\mathscr{C}})$. Further $|\mathfrak{A}| = \emptyset$ iff $\mathfrak{A} \vDash (\forall x)(x \neq x)$ and thus we may denote the last sentence by θ_0.

Let $A \in \mathscr{A}_k$ and denote the following formula by $\delta_A^e(x)$:

$$\theta_k(x) \wedge \bigwedge_{(C,f)} R_C(x \cdot f) \wedge \bigwedge_{(D,g)} \neg R_D(x \cdot g)$$

where $x \in V^k$, (C,f) ranges over all structured maps $f: |C| \to k$ such that $C \in \mathscr{C}_k$, and $f: C \to A$ is a morphism and (D,g) over all structured maps $g: |D| \to k$ such that $D \in \mathscr{C}_k$ and g is not a morphism $D \to A$. Hence δ_A^e is a conjunction of a subset of the elementary diagram Δ_A^e of the model $G_{\mathscr{C}}(A)$.

We will prove that for any $X \in \mathscr{A}$ and $a \in |X|^k$ holds

$$X \vDash \delta_A^e[a] \quad \text{iff } a: A \to X \text{ is an isomorphism.} \tag{3.2}$$

Let $X \models \delta_A^e[a]$. Then a is a bijective homomorphism $A \to X$. Assume that a^{-1} is not a homomorphism $X \to A$. Then there is $D \in \mathscr{C}_k$ and $g: D \to X$ such that $a^{-1} \cdot g$ is not a homomorphism $D \to A$. Hence $X \models \neg R_C[a \cdot (a^{-1} \cdot g)]$, which is a contradiction. Therefore a is an isomorphism $A \to X$. The reverse implication is easy.

(3.2) implies that the following sentences belong to $T_{\mathscr{C}}$:

$$\varphi_k = \left[(\exists x)\theta_k(x) \Rightarrow (\exists z) \bigvee_{A \in \mathscr{A}_k} \delta_A^e(z) \right],$$

for $k \in \mathrm{Card}$ and $x, z \in V^k$. Since $\mathscr{C}$ is strongly finally dense, the following sentences ψ_h belong to $T_{\mathscr{C}}$ for any morphism $h: D \to A$, $D \in \mathscr{C}$, $A \in \mathscr{A}_k$ and $k \in \mathrm{Card}$:

$$\psi_h = (\forall x) \left[\bigwedge_{(C,f)} R_C(x \cdot f) \Rightarrow R_D(x \cdot h) \right].$$

Here $x \in V^k$ and (C, f) ranges over all structured maps $f: |C| \to k$ such that $C \in \mathscr{C}_k$ and $f: C \to A$ is a morphism.

Lastly, (3.2) implies that the following sentences χ_g belong to $T_{\mathscr{C}}$ for any $g: |D| \to |A|$, $D \in \mathscr{C}$, $A \in \mathscr{A}_k$, $k \in \mathrm{Card}$, such that g is not a morphism $D \to A$:

$$\chi_g = (\forall x)[\delta_A^e(x) \Rightarrow \neg R_D(x \cdot g)].$$

Come back to our starting model $\mathfrak{B}$. Since $\mathfrak{B} \models (\exists x)(\theta_n(x)) \wedge \varphi_n$, there is $B \in \mathscr{A}_n$ such that $\mathfrak{B} \models (\exists z)\delta_B^e(z)$. Thus there is $b \in n^n$ such that $\mathfrak{B} \models \delta_B^e[b]$. Hence b is a bijection and $\mathfrak{B} \models \bigwedge_{(C,f)} R_C[b \cdot f]$. Thus, using ψ_h, $\mathfrak{B} \models R_D[b \cdot h]$ for any $h: D \to B$, $D \in \mathscr{C}$. Therefore $b: B \to \mathfrak{B}$ is a homomorphism.

Assume that there is $a \in n^{|D|}$, $D \in \mathscr{C}$ such that $\mathfrak{B} \models R_D[a]$ and $B \models \neg R_D[b^{-1} \cdot a]$. Then $\mathfrak{B} \models \chi_{b^{-1} \cdot a}$ and thus $\mathfrak{B} \models \neg R_D[b \cdot (b^{-1} \cdot a)]$, which is a contradiction. Hence $b^{-1}: \mathfrak{B} \to B$ is a homomorphism and therefore $\mathfrak{B} \cong B$.

Let us remark that sentences making $G_{\mathscr{C}}$ an isomorphism were explicitly described during the proof.

3.7. Corollary. *Any fibre-small concrete category is isomorphic to a category of all models of an $\exists\forall$-theory.*

Proof. It follows by Lemma 3.2 and Proposition 3.6. Indeed, only $\exists\forall$-sentences were used in the proof of Proposition 3.6.

There is a concrete category which is not isomorphic to any $\mathrm{Mod}(T)$.

3.8. Example. Let us consider the ordered class Ord of all ordinals as a category (i.e. there is a unique morphism $a \to b$ iff $a \leq b$). Assign to each ordinal a the one-element set 1 and to each morphism of Ord the identity mapping on 1. We show that the resulting concrete category $\mathscr{A}$ is not isomorphic to any $\mathrm{Mod}(T)$.

Assume that there is a theory T of type τ and an isomorphism $F: \mathscr{A} \to \mathrm{Mod}(T)$.

Since T has only one-element models, without loss of generality we may assume that τ has no function symbols. Also, we can suppose that τ has 0-ary relation symbols only (we may associate to each relation symbol R a 0-ary relation symbol R_0 such that $\mathfrak{A} \models R_0$ iff $\mathfrak{A} \models (\exists x)R(x)$ for any model $\mathfrak{A}$ of T).

Assign to each (i.e. 0-ary) relation symbol R the least ordinal a_R such that $F(a_R) \models R$. If $a \geq a_R$ then $F(a) \models R$ because the identity on 1 is a homomorphism $F(a_R) \to F(a)$ of models. Hence $F(a) \models R$ iff $a \geq a_R$.

If $a \in \mathrm{Ord}$ then there is $R \in \mathrm{Rel}(\tau)$ such that $a_R > a$. Otherwise the T-models $F(a)$, $F(b)$ would be isomorphic for any $b > a$.

Consider the τ-model $\mathfrak{A}$ such that $|\mathfrak{A}| = 1$ and $\mathfrak{A} \models R$ for any $R \in \mathrm{Rel}(\tau)$. We have shown that $\mathfrak{A}$ cannot be isomorphic to any model from $F(\mathscr{A})$. We will get a contradiction if we show that $\mathfrak{A}$ is a model of T. But any sentence $\varphi \in T$ contains only a set of relation symbols and thus there is an ordinal a such that $a > a_R$ for any relation symbol R occuring in φ. Hence $F(a) \models R$ iff $\mathfrak{A} \models R$ for any such R. Since $F(a) \models \varphi$, we get that $\mathfrak{A} \models \varphi$.

The proof of Proposition 3.6 refines the method of diagrams which is used e.g. in 1.6 of [16] and which immediately yields the following result.

3.9. Theorem. *Any concrete category is isomorphic to the category* Mod(T) *of all models of a theory T of some language $L_{\infty^*,\infty}(\tau)$.*

Proof. It suffices to replace δ^e_A from the proof of Proposition 3.6 by the conjunction of all formulas from the elementary diagram δ^e_A (or, more economically, to let C and D range over all $\mathscr{A}_*$).

Any sentence

$$(\forall x)(R_C(x \cdot f) \Leftrightarrow R_D(x \cdot g)), \tag{3.3}$$

where $f : |C| \to n$, $g : |D| \to n$ are two equivalent structured maps into n and $x \in V^n$, belongs to $T_{\mathscr{C}}$.

3.10. Lemma. *Let $\mathscr{A}$ be a strongly fibre-small concrete category, $\mathscr{C} \subseteq \mathscr{A}_*$ and $T \subseteq T_{\mathscr{C}}$ contain sentences* (3.3). *Then T is normal.*

Proof. If $x \in V^n$ then atomic formulas $\varphi(x)$ of $\tau_{\mathscr{C}}$ are of the form $R_C(x \cdot f)$ where $C \in \mathscr{C}$ and $f : |C| \to n$. Since T contains sentences (3.3), T is normal because $\mathscr{A}$ is strongly fibre-small.

3.11. Theorem. *A concrete category $\mathscr{A}$ is isomorphic to* Mod(T) *for a normal theory T iff $\mathscr{A}$ is strongly fibre-small.*

Proof. This follows by Corollary 2.6, Proposition 3.6 and Lemma 3.10.

3.12. Remark. We may replace in all preceding considerations sets by finite sets and classes by countable sets. Then the language $L_{\infty,\infty}$ is replaced by $L_{\omega,\omega}$ and $L_{\infty^{\cdot},\infty}$ by $L_{\omega_1,\omega}$. Then Theorems 3.9 and 3.7 yield the following results where a concrete category $\mathscr{A}$ over Fin means a concrete category $\mathscr{A}$ such that $|A|$ is finite for any $A \in \mathscr{A}$.

Any countable concrete category $\mathscr{A}$ over Fin *is isomorphic to a category of all finite models of some theory of a countable language $L_{\omega_1,\omega}(\tau)$.*

Any countable fibre-finite concrete category over Fin *is isomorphic to a category of all finite models of some theory of a countable language $L_{\omega,\omega}(\tau)$.*

Fibre-finiteness means here that $\mathscr{A}_n$ is finite for any natural number n. The last assertion has the following well-known result about fine spectra of first-order theories as a consequence (see Taylor [3, p. 279]): For any function $f: \omega \to \omega$ there is a countable set of first-order sentences having exactly $f(n)$ non-isomorphic models with n elements for each natural number n.

Similarly we can treat Theorem 3.11.

4. Theories with isomorphic concrete categories of models

The previous considerations induce us to put the question: what can be said about theories with isomorphic concrete categories of models?

Let T be a theory of type τ. Let a type σ be an expansion of τ. Choose for any cardinal $n>0$ and any $R \in \mathrm{Rel}_n(\sigma) - \mathrm{Rel}_n(\tau)$ an existential-positive formula $\varphi_R(x)$ of type τ where $x \in V^n$. Assign to any propositional variable $R \in \mathrm{Rel}_0(\sigma) - \mathrm{Rel}_0(\tau)$ an existential-positive sentence φ_R of type τ or the sentence $\varphi_R = (\exists x)(x \neq x)$. Choose similarly for any cardinal n and any $f \in \mathrm{Fnt}_n(\sigma) - \mathrm{Fnt}_n(\tau)$ an existential-positive formula $\varphi_f(y,x)$ of type τ, where $y \in V$ and $x \in V^n$, such that $T \models (\forall x)(\exists! y)\varphi_f(y,x)$. Let the theory S of type σ arise by adding the following sentences to T (where R and f run over relation and function symbols of σ not belonging to τ):

$$(\forall x)(R(x) \Leftrightarrow \varphi_R(x)), \qquad (\forall x)(\forall y)(y = f(x) \Leftrightarrow \varphi_f(y,x)). \tag{4.1}$$

Then we say that S is a *coherent extension* of T.

4.1. Lemma. *Let S be a coherent extension of T. Then* $\mathrm{Mod}(T) \cong \mathrm{Mod}(S)$.

Proof. Let $F: \mathrm{Mod}(S) \to \mathrm{Mod}(T)$ be the functor which carries each S-model to its τ-reduct. Assign to a T-model $\mathfrak{A}$ the following σ-model $H(\mathfrak{A})$: $|H(\mathfrak{A})| = |\mathfrak{A}|$, $a \in R_{H(\mathfrak{A})}$ iff $\mathfrak{A} \models \varphi_R[a]$ for any $n \in \mathrm{Card}$, $R \in \mathrm{Rel}_n(\sigma) - \mathrm{Rel}_n(\tau)$ and $a \in |\mathfrak{A}|^n$, $R_{H(\mathfrak{A})} = R_{\mathfrak{A}}$ for any $R \in \mathrm{Rel}(\tau)$, $b = f_{H(\mathfrak{A})}(a)$ iff $\mathfrak{A} \models \varphi_f[b,a]$ for any $n \in \mathrm{Card}$, $f \in \mathrm{Fnt}_n(\sigma) - \mathrm{Fnt}_n(\tau)$, $a \in |\mathfrak{A}|^n$ and $b \in |\mathfrak{A}|$, and finally $f_{H(\mathfrak{A})} = f_{\mathfrak{A}}$ for any $f \in \mathrm{Fnt}(\tau)$. Since existential-positive formulas are preserved under homomorphisms (and the same holds for the sentence $(\exists x)(x \neq x)$), $H: \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ is a functor.

Clearly $F \cdot H = 1$. Let $\mathfrak{B} \in \mathrm{Mod}(S)$, $R \in \mathrm{Rel}_n(\sigma) - \mathrm{Rel}_n(\tau)$ and $a \in |\mathfrak{B}|^n$. Following (4.1), $\mathfrak{B} \models R[a]$ iff $\mathfrak{B} \models \varphi_R[a]$. But it is equivalent with $F(\mathfrak{B}) \models \varphi_R[a]$ and thus with $H \cdot F(\mathfrak{B}) \models R[a]$. If $R \in \mathrm{Rel}_n(\tau)$ then $\mathfrak{B} \models R[a]$ iff $F(\mathfrak{B}) \models R[a]$ and it is equivalent with $H \cdot F(\mathfrak{B}) \models R[a]$. Analogously for function symbols. Therefore $H \cdot F = 1$ and the proof is complete.

Two theories T and S (generally of distinct types) are said to be *coherently equivalent* if some coherent extension of T is equivalent to some coherent extension of S.

Clearly any *equivalent* theories (i.e. theories of the same type with the same models) are coherently equivalent. Coherent extensions are a special case of a standard concept of an extension by means of definitions and thus the coherent equivalence is an instance of weak equivalence (see e.g. [30, Ch. 4, ex. 2]).

4.2. Corollary. *Coherently equivalent theories have isomorphic concrete metacategories of models.*

Exploiting an example invented by Reiterman [25] we show that Corollary 4.2 cannot be reversed.

4.3. Example. Let τ consist of a constant 0, a binary function symbol $+$ and of unary function symbols f_{ik} where i carries over ordinals and k over regular cardinals such that $i<k$. Let T be an equational theory of $L_{\omega,\omega}(\tau)$ given by sentences:

$$(\forall x)(x+x=0), \qquad (\forall x)(x+0=0),$$
$$(\forall x)(f_{ik}(x)+f_{jk}(x)=f_{ik}(x)) \quad \text{for } i<j.$$

It is shown in [25] that $\mathrm{Mod}(T)$ is fibre-small. Following Corollary 3.7, $\mathrm{Mod}(T) \cong \mathrm{Mod}(T_*)$ where T_* is the canonical theory of $\mathrm{Mod}(T)$ with respect to $\mathrm{Mod}(T)_*$. The canonical type of $\mathrm{Mod}(T)$ will be denoted by τ_*.

Assume that T and T_* are coherently equivalent. Then there is a type $\sigma \supseteq \tau, \tau_*$ and equivalent theories S_1, S_2 of type σ such that S_1 is a coherent extension of T and S_2 of T_*. Denote by $H_1: \mathrm{Mod}(T) \to \mathrm{Mod}(S_1)$ the functor H from the proof of Lemma 4.1. Consider the relation symbol $R = R_{\mathfrak{C}} \in \tau_*$ where $|\mathfrak{C}| = \{0\}$. There is an existential-positive formula $\varphi_R(x)$ of type τ such that $S_1 \models (\forall x)(R(x) \Leftrightarrow \varphi_R(x))$. There is an infinite cardinal n such that whenever f_{ik} occurs in φ_R then $k<n$.

Consider the τ-algebra $\mathfrak{A}$ such that $|\mathfrak{A}| = n$, $0_{\mathfrak{A}} = 0$, $a +_{\mathfrak{A}} b = \min\{a,b\}$ for $a \neq b$ and $a +_{\mathfrak{A}} a = 0$, $(f_{ik})_{\mathfrak{A}}(a) = 0$ for $k \neq n$ and $(f_{ik})_{\mathfrak{A}}(a) = a+i$ (see [25]). Let $\psi(y)$, $y \in V^m$, be a quantifier-free positive formula which does not contain any f_{ik} with $k \geq n$. Then $\mathfrak{A} \models \psi[\bar{0}]$ where $\bar{0} \in n^m$ is constant with the value 0. Hence $\mathfrak{A} \models \varphi_R[0]$ and thus $H_1(\mathfrak{A}) \models R[0]$.

It holds $T_* \models (\forall x)(R(x) \Rightarrow Q(\bar{x}))$ where Q is an arbitrary relation symbol of τ_* and $\bar{x}$ is the constant mapping into V with the value x. Hence $S_1 \models (\forall x)(R(x) \Rightarrow Q(\bar{x}))$ and therefore

$$H_1(\mathfrak{A}) \models Q[\bar{0}] \quad \text{for any } Q \in \mathrm{Rel}(\tau_*). \tag{4.2}$$

There is an existential-positive formula $\varphi_f(y,x)$ of type τ_* such that $S_2 \models (\forall x)(\forall y)(y = f(x) \Leftrightarrow \varphi_f(y,x)$ where $f = f_{1n}$. Hence $S_1 \models (\forall x)(\forall y)(y = f(x) \Leftrightarrow \varphi_f(y,x))$. Since φ_f is existential-positive, $H_1(\mathfrak{A}) \models \varphi_f[0,0]$ by (4.2). Thus $H_1(\mathfrak{A}) \models f_{1n}(0) = 0$, which is a contradiction.

Nevertheless, there is a great class of theories for which the converse of Corollary 4.2 can be proved.

A formula α is said to be a *basic Horn formula* if α is the disjunction $\bigvee_{i \in n} \theta_i$, where at most one of the formulas θ_i is an atomic formula, the rest being negations of atomic formulas. Thus there are basic Horn formulas of two kinds:

$$\varphi \Rightarrow \psi \tag{4.3}$$

where φ is a $\bigwedge$-formula and ψ is atomic and

$$\neg \varphi \tag{4.4}$$

where φ is atomic. Basic Horn formulas of type (4.3) are called *strict basic*. A (*strict*) *Horn formula* is built up from (strict) basic Horn formulas with the connectives $\bigwedge$, $\exists$ and $\forall$. A (*strict*) *Horn sentence* is a (strict) Horn formula with no free variables. A theory equivalent to a theory consisting of (strict) Horn sentences is called a (*strict*) *Horn theory*. If ψ in (4.3) is not the equality of two terms then (4.3) will be called a *strict basic Horn formula without equality*. The concept of a *strict Horn theory without equality* is now clear.

Let a theory T consist of sentences

$$(\forall x)(\neg \varphi(x)) \tag{4.5}$$

where φ is atomic and of sentences

$$(\forall x)(\exists y)(\varphi(x) \Rightarrow \psi(x,y)) \tag{4.6}$$

where φ and ψ are $\bigwedge$-formulas such that

$$T \models (\forall x)[\varphi(x) \Rightarrow (\exists! y)\psi(x,y)]. \tag{4.7}$$

A theory equivalent to a theory T of the above kind is called a *$\exists!$-theory*. A $\exists!$-theory is said to be *strict* if it does not contain sentences (4.5).

Strict $\exists!$-theories form an infinitary version of one-sorted $\exists!$-theories of M. Coste (see [7], also [8] where the term $\underrightarrow{\lim}$-theories is used). Since any sentence (4.6) (with $\psi(x,y) = \bigwedge_{i \in I} \psi_i(x,y)$) is equivalent to the sentence $(\forall x)(\exists y)\bigwedge_{i \in I}(\varphi(x) \Rightarrow \psi_i(x,y))$, (strict) $\exists!$-theories are instances of universal-existential (strict) Horn theories. Since any universal strict Horn sentence $(\forall x)(\varphi(x) \Rightarrow \psi(x))$ is equivalent to the sentence $(\forall x)(\exists y)(\varphi(x) \Rightarrow \psi(x) \wedge y = x)$, which satisfies (4.7), any universal (strict) Horn theory is a (strict) $\exists!$-theory; (in fact, $\exists!$-theories are precisely universal Horn theories with partial function symbols allowed).

A theory T will be called 0-*tight* if for each $x \in V$ the class of all $\exists\wedge$-formulas $\varphi(x)$ such that $T \models (\exists! x)\varphi(x)$ has a T-representative set.

4.4. Lemma. *T is 0-tight whenever* Mod(T) *has an initial object.*

Proof. Let Mod(T) have an initial object $\mathfrak{A}$. Consider two $\exists\wedge$-formulas $\varphi(x)$ and $\psi(x)$ such that $T \models (\exists! x)\varphi(x)$, $T \models (\exists! x)\psi(x)$ and $\mathfrak{A} \models \varphi[a] \wedge \psi[a]$. Let $\mathfrak{B}$ be a T-model and f a unique homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then $\mathfrak{B} \models \varphi[f(a)] \wedge \psi[f(a)]$ and thus $\mathfrak{B} \models (\forall x)(\varphi(x) \Leftrightarrow \psi(x))$. Hence $\varphi \sim_T \psi$. Therefore T is 0-tight.

The last lemma may be reversed for $\exists!$-theories, which is an infinitary version of a result of Coste (see [7, p. 4] or [8, III.1.1]).

4.5. Proposition. *Let T be a $\exists!$-theory. Then* Mod(T) *has an initial object iff T is 0-tight and has a model.*

Proof. Let T be a 0-tight $\exists!$-theory having a model. We may assume that T consists of sentences (4.5) and (4.6).

Choose $x \in V$ and denote by A a T-representative set of $\exists\wedge$-formulas $\varphi(x)$ such that $T \models (\exists! x)\varphi(x)$. Remark that elements $a \in A^n$ correspond to formulas $a(y)$ where $y \in V^n$ and $T \models (\exists! y)a(y)$. This correspondence is given by assigning $a(y) = \bigwedge_{i\in n} a_i(y_i)$ to any n-tuple $a_i(x) \in A$, $i \in n$, and on the other hand by assigning $a_i(x) = (\exists y)(a(y) \wedge x = y_i)$ to each $a(y)$.

Define the τ-model $\mathfrak{A}$ on A as follows. If $f \in \mathrm{Fnt}_n(\tau)$ and $a \in A^n$ then $f_{\mathfrak{A}}(a) \in A$ is T-equivalent to the formula

$$(\exists y)(x = f(y) \wedge a(y)). \tag{4.8}$$

If $R \in \mathrm{Rel}_n(\tau)$ and $a \in A^n$ then the relation $R_{\mathfrak{A}} \subseteq A^n$ contains all $a \in A^n$ such that

$$T \models (\exists x)(R(x) \wedge a(x)). \tag{4.9}$$

A straightforward inductive proof on the 'complexity' of terms shows that

$$t_{\mathfrak{A}}[a](x) \sim_T (\exists y)(x = t(y) \wedge a(y))$$

for any term $t(x)$, $x \in V^n$, and any $a \in A^n$. Similarly, for any $\wedge$-formula $\alpha(x)$, $x \in V^n$, and $a \in A^n$ it holds that $\mathfrak{A} \models \alpha[a]$ iff

$$T \models (\exists x)(\alpha(x) \wedge a(x)). \tag{4.10}$$

We are now able to prove that $\mathfrak{A}$ is an initial object in Mod(T). Let $\mathfrak{B}$ be a T-model. Assign to each $\varphi \in A$ the unique element $g(\varphi) \in |\mathfrak{B}|$ such that $\mathfrak{B} \models \varphi[g(\varphi)]$. Let $f \in \mathrm{Fnt}_n(\tau)$ and $a \in A^n$. Since $\mathfrak{B} \models a[g^n(a)]$, following (4.8) $\mathfrak{B} \models f_{\mathfrak{A}}[a](f_{\mathfrak{B}}[g^n(a)])$.

Hence $g(f_{\mathfrak{A}}[a]) = f_{\mathfrak{B}}[g^n(a)]$. Let $R \in \mathrm{Rel}_n(\tau)$ and $a \in A^n$ such that $\mathfrak{A} \models R[a]$. Following (4.9), $T \models (\exists x)(R(x) \wedge a(x)$ and thus $\mathfrak{B} \models R[g^n(a)]$. Hence $g : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism. It follows moreover that $\mathfrak{A}$ satisfies any sentence (4.5) because T has a model.

Consider a sentence (4.6) of T. Let $a \in A^n$ such that $\mathfrak{A} \models \varphi[a]$. Consider $b(y) = (\exists x)(\varphi(x) \wedge \psi(x,y) \wedge a(x)) \in A^m$. Following (4.10), $T \models (\exists x)(\varphi(x) \wedge a(x))$ and by (4.6) $T \models (\exists x)(\exists y)(\varphi(x) \wedge \psi(x,y) \wedge a(x))$. Hence $T \models (\exists x)(\exists y)(\psi(x,y) \wedge a(x) \wedge b(y))$ and by (4.10) $\mathfrak{A} \models \psi[a,b]$. Hence $\mathfrak{A}$ is a T-model. Moreover, for any $\varphi \in A$ holds

$$\mathfrak{A} \models \varphi[\varphi] \tag{4.11}$$

Indeed, there is a unique $a \in A$ such that $\mathfrak{A} \models \varphi[a]$. Further, $\varphi = (\exists y)\psi(x,y)$ where $\psi(x,y)$ is a $\wedge$-formula and $y \in V^m$. Thus there is $b \in A^m$ such that $\mathfrak{A} \models \psi[a,b]$. Following (4.10) $T \models (\exists x)(\exists y)(\psi(x,y) \wedge a(x) \wedge b(y))$. Thus $T \models (\exists x)(\varphi(x) \wedge a(x))$, which implies that $\varphi \sim_T a$ (see the proof of Lemma 4.4). Therefore $a = \varphi$.

It remains to prove that the homomorphism $g : \mathfrak{A} \to \mathfrak{B}$ constructed above is unique. Let $h : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism and $\varphi \in A$. Following (4.11), $\mathfrak{A} \models \varphi[\varphi]$ and since φ is an $\exists\wedge$-formula, $\mathfrak{B} \models \varphi[h(\varphi)]$ holds. Hence $h = g$.

A theory T will be called *tight* if for any cardinal n the class of all $\exists\wedge$-formulas $\varphi(x,y)$ where $x \in V^n$, $y \in V$ and $T \models (\forall x)(\exists! y)\varphi(x,y)$ has a T-representative set.

Let T be a theory of type τ and $\varrho(x)$ a formula of type τ where $x \in V^n$. Two formulas $\varphi(x,y)$ and $\psi(x,y)$ will be called *ϱ-equivalent* if $T \models (\forall x)(\forall y)[\varrho(x) \Rightarrow (\varphi(x,y) \Leftrightarrow \psi(x,y)]$.

Denote by $T(\varrho)$ the theory $T \cup \{(\forall x)\varrho(x)\}$.

A formula ϱ will be called *diagrammatic* if $\varrho = \bigwedge_{i \in I} \varrho_i$ where each ϱ_i is either a formula $y = f(x)$ where $f \in \mathrm{Fnt}(\tau)$ or a formula $R(x)$ with $R \in \mathrm{Rel}(\tau)$. Hence diagrammatic formulas are special kinds of $\wedge$-formulas.

A theory T will be called *strongly tight* if for any diagrammatic formula $\varrho(x)$ the class of all $\exists\wedge$-formulas $\varphi(x,y)$ such that $y \in V$ and

$$T \models (\forall x)(\varrho(x) \Rightarrow (\exists! y)\varphi(x,y)) \tag{4.12}$$

has a ϱ-representative set.

Clearly any theory of a small type is strongly tight and any strongly tight theory is tight (put $\varrho(x) = (x = x)$).

If a type σ is a simple expansion of τ and T a theory of type τ then T_σ will denote T considered as a theory of type σ. If $\varrho(x)$ is a formula of type τ, $x \in V^n$ and σ is a simple expansion by means of constants c_i, $i \in n$, then $\varrho(c)$ will denote the sentence obtained from $\varrho(x)$ by substituting c_i for x_i for any $i \in n$.

4.6. Lemma. *A theory T is strongly tight iff $T(\varrho(c))$ is* 0-*tight for any diagrammatic formula ϱ. In particular, T is tight iff T is* 0-*tight for any simple expansion σ of τ.*

Proof. Evident.

Let us assign to a type τ a new type $\bar{\tau}$ which has the same relation symbols as τ and each $f\in \mathrm{Fnt}_n(\tau)$ is replaced by $R_f\in \mathrm{Rel}_{n+1}(\bar{\tau})$. Denote by $U:\mathrm{Mod}(\tau)\to\mathrm{Mod}(\bar{\tau})$ the evident functor which interprets an n-ary function as the corresponding $n+1$-ary relation, i.e. having $\mathfrak{A}\in\mathrm{Mod}(\tau)$ then $|U(\mathfrak{A})|=|\mathfrak{A}|$, $R_{U(\mathfrak{A})}=R_{\mathfrak{A}}$ for any $R\in\mathrm{Rel}(\tau)$ and $(R_f)_{U(\mathfrak{A})}=\{(b,a)\mid b\in|\mathfrak{A}|,\ a\in|\mathfrak{A}|^n \text{ and } b=f_{\mathfrak{A}}[a]\}$ for any $f\in\mathrm{Fnt}_n(\tau)$.

4.7. Proposition. *A normal strict $\exists!$-theory T of type τ is strongly tight iff* $U:\mathrm{Mod}(T)\to\mathrm{Mod}(\bar{\tau})$ *has a left adjoint.*

Proof. Assume that T is a strongly tight normal strict $\exists!$-theory and let $\mathfrak{A}\in\mathrm{Mod}(\bar{\tau})$. Consider $\Delta=\{R(x)\mid R(x)\in\Delta_{\mathfrak{A}}^+\}\cup\{y=f(x)\mid R_f(y,x)\in\Delta_{\mathfrak{A}}^+\}$. Since T is normal, Δ has a T-representative set $\bar{\Delta}$. Let ϱ be the conjunction of formulas from $\bar{\Delta}$. Hence $f:\mathfrak{A}\to U(\mathfrak{B})$ is a homomorphism iff $\mathfrak{B}\models\varrho[f]$ for any $\mathfrak{B}\in\mathrm{Mod}(T)$ and $f:|\mathfrak{A}|\to|\mathfrak{B}|$. Denote by σ the simple expansion by means of constants c_a, $a\in|\mathfrak{A}|$. Models $\mathfrak{B}$ of the theory $T(\varrho(c))$ correspond to homomorphisms $\mathfrak{A}\to U(\mathfrak{B})$ with $\mathfrak{B}\in\mathrm{Mod}(T)$.

If $\mathfrak{B}_0$ is a one-element model such that $R_{\mathfrak{B}_0}=|\mathfrak{B}_0|^n$ for any $R\in\mathrm{Rel}(\tau)$, then a unique mapping $\mathfrak{A}\to\mathfrak{B}_0$ is a homomorphism. Hence $T(\varrho(c))$ has a model and by Proposition 4.5 and Lemma 4.6 it has an initial model $\mathfrak{A}\to UF(\mathfrak{A})$. It gives the value of a left adjoint F to U and the unit of the adjunction.

The converse follows from the fact that any diagrammatic formula of type τ arises from the positive diagram of some $\bar{\tau}$-model.

The functor U and its left adjoint were considered by Nelson [22] in a special case.

4.8. Corollary. *Let T be a normal strongly tight strict $\exists!$-theory of type τ. Then the forgetful functor* $\mathrm{Mod}(T)\to\mathrm{Mod}(\sigma)$ *has a left adjoint for any reduction σ of τ.*

4.9. Corollary. *Let T be a $\exists!$-theory. Then T is tight iff* $\mathrm{Mod}(T)$ *has free objects.*

Let us emphasize that the underlying set of a free T-model over a cardinal n is a T-representative set of $\exists\wedge$-formulas $\varphi(x,y)$ where $x\in V^n$, $y\in V$ and $T\models(\forall x)(\exists!y)\varphi(x,y)$.

4.10. Theorem. *Let S and T be normal strongly tight strict $\exists!$-theories. Then* $\mathrm{Mod}(S)\cong\mathrm{Mod}(T)$ *iff S and T are coherently equivalent.*

Proof. Let S or T be normal strongly tight strict $\exists!$-theories of type σ or τ respectively such that there is an isomorphism $H:\mathrm{Mod}(S)\to\mathrm{Mod}(T)$. We have to show that S and T are coherently equivalent.

Denote by F_S and F_T left adjoints to the underlying set functors which are given by Corollary 4.9 and by $\psi_S:\mathrm{Mod}(S)(F_S(X),\mathfrak{A})\to\mathrm{Set}(X,|\mathfrak{A}|)$, ψ_T the adjunction isomorphisms. There is a natural isomorphism $\zeta:F_T\to H\cdot F_S$ such that

$$|(\psi_T^{-1})_{X,H(\mathfrak{A})}(a)|=|(\psi_S^{-1})_{X,\mathfrak{A}}(a)|\cdot|\zeta_X|$$

for any set X, $\mathfrak{A}\in \mathrm{Mod}(S)$ and $a\in|\mathfrak{A}|^X$. Since the formula $y=f(x)$ belongs to $|F_T(n)|$ for a cardinal n, $x\in V^n$ and $f\in \mathrm{Fnt}_n(\tau)$, we may consider the formula $\varphi_f(y,x)=(\zeta_n)(y=f(x))$ of type σ.

Let $R\in \mathrm{Rel}_n(\tau)$. Consider the $\bar\tau$-model $\mathfrak{A}_R$ such that $|\mathfrak{A}_R|=n$, $R_{\mathfrak{A}}=\{1_n\}$ and $Q_{\mathfrak{A}}=\emptyset$ for any $R\neq Q\in \mathrm{Rel}(\bar\tau)$. We know by Proposition 4.7 that $U:\mathrm{Mod}(T)\to \mathrm{Mod}(\bar\tau)$ has a left adjoint F. Denote by $\psi:\mathrm{Mod}(T)(F(\mathfrak{A}),\mathfrak{B})\to \mathrm{Mod}(\bar\tau)(\mathfrak{A},U(\mathfrak{B}))$ the adjunction isomorphism and by η the unit of the adjunction. Put $\mathfrak{B}_R=H^{-1}\cdot F(\mathfrak{A}_R)$ and $\varphi_R(x)=(\exists z)\ \delta^+_{\mathfrak{B}_R}(z)\wedge z\cdot \eta_{\mathfrak{A}_R}=x$.

Let π be the disjoint union of types τ and σ, $\bar S$ be the coherent extension of S of type π given by the above formulas φ_f, φ_R, and $\bar T$ the analogous coherent extension of T. We want to show that the theories $\bar S$ and $\bar T$ are equivalent.

Assign to any formula $\alpha(x)$ of type τ a formula $\bar\alpha(x)$ of type σ as follows: If $f\in \mathrm{Fnt}(\tau)$ then $\overline{y=f(x)}$ equals to $\varphi_f(y,x)$. Inductively, if $f\in \mathrm{Fnt}_n(\tau)$ and t_i, $i\in n$, are terms of type τ, then

$$y=\overline{f(t_1(x),\dots,t_i(x),\dots)}=(\exists z)\left(\varphi_f(y,z)\wedge \bigwedge_{i\in n}\overline{z_i=t_i(x)}\right).$$

Further,

$$\overline{R(t_1(x),\dots,t_i(x),\dots)}=(\exists z)\left(\varphi_R(z)\wedge \bigwedge_{i\in n}\overline{z_i=t_i(x)}\right)$$

for $R\in \mathrm{Rel}_n(\tau)$ and terms t_i, $i\in n$, of type τ. Finally, $\overline{\bigwedge_{i\in I}\alpha_i}=\bigwedge_{i\in I}\overline{\alpha_i}$, $\overline{\neg\alpha}=\neg\bar\alpha$ and $\overline{(\exists x)\alpha}=(\exists x)\bar\alpha$. We may analogously assign to each formula β of type σ a formula $\bar\beta$ of type τ.

We are going to show that

$$\mathfrak{A}\models\alpha[a]\Leftrightarrow H^{-1}(\mathfrak{A})\models\bar\alpha[a] \tag{4.13}$$

holds for any formula α of type τ, any $\mathfrak{A}\in \mathrm{Mod}(T)$, and any $a\in|\mathfrak{A}|^n$. Let $\alpha=(y=f(x))$. Since

$$\begin{aligned}f_{\mathfrak{A}}[a]&=|(\varphi_T^{-1})_{n,\mathfrak{A}}(a)|(\alpha)=|(\varphi_S^{-1})_{n,H^{-1}(\mathfrak{A})}(a)|\cdot|\zeta_n|(\alpha)\\&=|(\varphi_S^{-1})_{n,H^{-1}(\mathfrak{A})}(a)|(\varphi_f(y,x)),\end{aligned}$$

we have $H^{-1}(\mathfrak{A})\models\varphi_f[f_{\mathfrak{A}}(a),a]$. Further $\mathfrak{A}\models R[a]$ iff $a:U(\mathfrak{A}_R)\to\mathfrak{A}$ is a homomorphism, i.e. iff there exists a homomorphism $b:F(\mathfrak{A}_R)\to\mathfrak{A}$ such that $a=b\cdot\eta_{\mathfrak{A}_R}$. However, b is a homomorphism $F(\mathfrak{A}_R)\to\mathfrak{A}$ iff it is a homomorphism $\mathfrak{B}_R\to H^{-1}(\mathfrak{A})$, i.e. iff $H^{-1}(\mathfrak{A})\models\delta^+_{\mathfrak{B}_R}[b]$. Hence $\mathfrak{A}\models R[a]$ iff $H^{-1}(\mathfrak{A})\models\varphi_R[a]$. Now, (4.13) follows by a straightforward induction on the complexity of formulas.

(4.13) immediately yields that $T\models\alpha$ iff $S\models\bar\alpha$ for any formula α of type τ and (4.13) together with the analogous assertion for $\bar\beta$ gives that $S\models(\beta\Leftrightarrow\bar{\bar\beta})$ for any formula β of type σ.

We may now prove that $\bar T\sim\bar S$. We have $\bar T\sim T_1=T\cup\{\beta\Leftrightarrow\bar\beta\mid\beta$ a formula of type $\sigma\}$ and analogously for $\bar S$. Let $\alpha\in T$. Then $S\models\bar\alpha$ and thus $\bar S\models\alpha$. If $\alpha=(\beta\Leftrightarrow\bar\beta)$, then $S\models(\beta\Leftrightarrow\bar{\bar\beta})$ and $(\bar{\bar\beta}\Leftrightarrow\bar\beta)\in S_1$, i.e. $\bar S\models\alpha$. The proof of $\bar S\models\beta\Rightarrow\bar T\models\beta$ is analogous.

To conclude this section we show that the converse of Corollary 4.2 holds for theories of the language $L_{\infty^+,\infty}$.

4.11. Theorem. *Let S and T be theories of $L_{\infty^+,\infty}$ such that Mod(S) and* Mod(T) *are categories. Then* Mod(S)$\cong$Mod(T) *iff S and T are coherently equivalent.*

Proof. Lemma 4.1 works for theories of $L_{\infty^+,\infty}$, too. Let S and T be theories of $L_{\infty^+,\infty}(\sigma)$ and $L_{\infty^+,\infty}(\tau)$ such that there is an isomorphism $H:\mathrm{Mod}(S)\to\mathrm{Mod}(T)$. Assign to any $f\in\mathrm{Fnt}_n(\tau)$ and $R\in\mathrm{Rel}_n(\tau)$ the following formulas of $L_{\infty^+,\infty}(\sigma)$:

$$\varphi_f(y,x)=\bigvee_{\mathfrak{B}\in\mathrm{Mod}(S)}(\exists z)\left[\bigwedge_{\alpha\in\Delta^+_{\mathfrak{B}}}\alpha(z)\wedge\bigvee_{d\in|\mathfrak{B}|^n}(x=z\cdot d\wedge y=z_{f_{H(\mathfrak{B})}[d]})\right]$$

$$\varphi_R(x)=\bigvee_{\mathfrak{B}\in\mathrm{Mod}(S)}(\exists z)\left(\bigwedge_{\alpha\in\Delta^+_{\mathfrak{B}}}\alpha(z)\wedge\bigvee_{d\in R_{H(\mathfrak{B})}}x=z\cdot d\right).$$

We prove that

$$b=f_{\mathfrak{A}}[a]\Leftrightarrow H^{-1}(\mathfrak{A})\models\varphi_f[b,a] \tag{4.14}$$

$$a\in R_{\mathfrak{A}}\Leftrightarrow H^{-1}(\mathfrak{A})\models\varphi_R[a] \tag{4.15}$$

for any $\mathfrak{A}\in\mathrm{Mod}(T)$, $f\in\mathrm{Fnt}_n(\tau)$, $R\in\mathrm{Rel}_n(\tau)$ and $a\in|\mathfrak{A}|^n$. We prove (4.15), the verification of (4.14) is analogous. If $a\in R_{\mathfrak{A}}$, then $H^{-1}(\mathfrak{A})\models\varphi_R[a]$ because $H^{-1}(\mathfrak{A})\models\bigwedge_{\alpha\in\Delta^+_{H^{-1}(\mathfrak{A})}}\alpha[1_{|\mathfrak{A}|}]$ holds. Let $H^{-1}(\mathfrak{A})\models\varphi_R[a]$. Then there is $\mathfrak{B}\in\mathrm{Mod}(S)$, $c\in|\mathfrak{A}|^{|\mathfrak{B}|}$ and $d\in R_{H(\mathfrak{B})}$ such that $H^{-1}(\mathfrak{A})\models\bigwedge_{\alpha\in\Delta^+_{\mathfrak{B}}}\alpha[c]\wedge a=c\cdot d$. Hence $c:\mathfrak{B}\to H^{-1}(\mathfrak{A})$ is a homomorphism and therefore $c:H(\mathfrak{B})\to\mathfrak{A}$ has to be a homomorphism. Thus $a\in R_{\mathfrak{A}}$.

The rest of the proof proceeds in the same way as (4.13) has given Theorem 4.10.

5. Initially complete categories

The notion of an initially complete category was recalled in Section 3. An *initial completion* of a concrete category $\mathscr{A}$ is an initially complete category containing $\mathscr{A}$ as a full subcategory. A *MacNeille completion* of $\mathscr{A}$ is an initial completion in which $\mathscr{A}$ is moreover initially and finally dense. A MacNeille completion of $\mathscr{A}$, if it exists, is unique (up to an isomorphism) and it is the smallest initial completion of $\mathscr{A}$ (see Herrlich [13] and Porst [23]).

A theory of a type having only relation symbols will be called *relational*.

5.1. Proposition. *Let T be a relational universal strict Horn theory without equality. Then* Mod(T) *is initially complete.*

Proof. Consider a nonempty source $(f_i:A\to|\mathfrak{A}|_i)_{i\in I}$ where $\mathfrak{A}_i$ are T-models. Define the τ-model $\mathfrak{A}$ on A as follows: $R_{\mathfrak{A}}=\{a\in A^n\mid\mathfrak{A}_i\models R[f_i\cdot a]$ for any $i\in I\}$ for any

$R \in \mathrm{Rel}_n(\tau)$. Since T is given by sentences of the following two kinds:

(1) $(\forall x)R(x \cdot g)$ where $R \in \mathrm{Rel}_n(\tau)$, $x \in V^n$ and $g \in n^k$,

(2) $(\forall x)(\bigwedge_{j \in J} R_j(x \cdot g_j) \rightarrow R(x \cdot g))$ where $R, R_j \in \mathrm{Rel}(\tau)$, $x \in V^n$, $g \in n^k$ and $g_j \in n^{k_j}$ for any $j \in J$,

it is easy to see that $\mathfrak{A}$ is a T-model. Hence $\mathfrak{A}$ is the desired initial lift of (f_i).

The existence of an initial lift of the empty source means the existence of a terminal object. However, the following τ-model $\mathfrak{A}_0$ is a terminal T-model: $|\mathfrak{A}_0| = 1$, $R_{\mathfrak{A}_0} = 1^n$ for any $R \in \mathrm{Rel}_n(\tau)$.

If $\mathscr{A}$ is a concrete category and $\mathscr{C} \subseteq \mathscr{A}_*$ then $T_{0,\mathscr{C}}$ will denote the theory of type $\tau_{\mathscr{C}}$ consisting of all universal strict Horn sentences without equality from $T_{\mathscr{C}}$.

Let $\mathfrak{A}$ be a $T_{0,\mathscr{C}}$-model. In what follows, we will denote by $P(\mathfrak{A})$ the sink consisting of all structured maps (C,p) such that $C \in \mathscr{C}$ and $p: C \rightarrow \mathfrak{A}$ is a homomorphism. If $\mathscr{C}$ is finally dense in $\mathscr{A}$ then Lemma 3.5 implies that $P(\mathfrak{A}) = \{(C,p) \mid C \in \mathscr{C},\ p: |C| \rightarrow |\mathfrak{A}| \text{ and } \mathfrak{A} \models R_C[p]\}$. If $\mathscr{A}$ is strongly fibre-small then $P(\mathfrak{A})$ has a $\sim$-representative set which will be denoted by $\bar{P}(\mathfrak{A})$. Hence if $\mathscr{C}$ is finally dense and $\mathscr{A}$ strongly fibre-small, then

$$\mathfrak{B} \models \bigwedge_{(C,p) \in \bar{P}(\mathfrak{A})} R_C[a \cdot p] \quad \text{iff } a: |\mathfrak{A}| \rightarrow |\mathfrak{B}| \text{ is a homomorphism.} \tag{5.1}$$

5.2. Theorem. *Let $\mathscr{A}$ be strongly fibre-small and $\mathscr{C} \subseteq \mathscr{A}_*$ be finally dense. Then* $\mathrm{Mod}(T_{0,\mathscr{C}})$ *is a MacNeille completion of $\mathscr{A}$.*

Proof. $\mathrm{Mod}(T_{0,\mathscr{C}})$ is initially complete by Proposition 5.1 and $\mathscr{A}$ is finally dense in it by Lemma 3.5. To show that $\mathscr{A}$ is initially dense in $\mathrm{Mod}(T_{0,\mathscr{C}})$ it suffices to prove that $h: |\mathfrak{A}| \rightarrow |\mathfrak{B}|$ is a homomorphism whenever $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}(T_{0,\mathscr{C}})$ and $f \cdot h: \mathfrak{A} \rightarrow A$ is a homomorphism for any homomorphism $f: \mathfrak{B} \rightarrow A$ and any $A \in \mathscr{A}$.

Consider thus such h and let $D \in \mathscr{C}$ and $b \in (R_D)_{\mathfrak{A}}$. Consider the sentence

$$\alpha = (\forall x)\left[\bigwedge_{(C,p) \in \bar{P}(\mathfrak{B})} R_C(x \cdot p) \Rightarrow R_D(x \cdot h \cdot b)\right].$$

We will check that $\alpha \in T$. Let $A \in \mathscr{A}$ and $a \in |A|^{|\mathfrak{B}|}$ such that $A \models \bigwedge_{(C,p) \in \bar{P}(\mathfrak{B})} R_C[a \cdot p]$. Then $a: \mathfrak{B} \rightarrow A$ is a homomorphism by (5.1). Hence $a \cdot h: \mathfrak{A} \rightarrow A$ is a homomorphism and thus $A \models R_D[a \cdot h \cdot b]$. Therefore $\alpha \in T$ and since $\mathfrak{B} \models \bigwedge_{(C,p) \in \bar{P}(\mathfrak{B})} R_C[p]$, we get that $\mathfrak{B} \models R_D[h \cdot b]$. Hence h is a homomorphism.

Since $\mathrm{Mod}(T_{0,\mathscr{C}})$ is fibre-small by Lemma 3.10 and Corollary 2.6, Theorem 5.2 provides the result of Adámek, Herrlich and Strecker asserting that $\mathscr{A}$ has a fibre-small MacNeille completion iff it is strongly fibre-small. It yields also the next theorem which was conjectured in [27].

Before stating it, we consider, as an illustration, the MacNeille completion of the category $\mathscr{A}$ of algebras with one unary operation. It is described in [1] as consisting of graphs (X, ϱ) subjected to the following conditions:

(I) For each $(x,y)\in\varrho$ there exists $z\in X$ with $(y,z)\in\varrho$.

(P) $(x,y)\in\varrho$ implies $(x',y')\in\varrho$ whenever $x\sim x'$ and $y\sim y'$; where $\sim$ is the least equivalence on X for which $(x,y_1)\in\varrho$, $(x,y_2)\in\varrho$ implies $y_1\sim y_2$.

(I) enables us to define ϱ by means of an ω-ary relation R given by

$$(\forall x)\left(R(x_1,x_2,\dots)\Leftrightarrow\bigwedge_{i=1}^{\infty}(x_i,x_{i+1})\in\varrho\right).$$

Now, $R=R_C$ where C is the free algebra on one generator, which is evidently finally dense in $\mathscr{A}$. (P) constitutes then the corresponding universal strict Horn theory without equality.

5.3. Theorem. *The following two conditions are equivalent for any concrete category* $\mathscr{A}$:

(i) $\mathscr{A}$ *is a fibre-small initially complete category.*

(ii) $\mathscr{A}\cong\mathrm{Mod}(T)$ *for a relational, normal, universal strict Horn theory T without equality.*

Proof. (ii)$\Rightarrow$(i) follows by Proposition 5.1 and Corollary 2.6. If $\mathscr{A}$ is initially complete then $\mathscr{A}$ is a MacNeille completion of itself and thus (i)$\Rightarrow$(ii) follows by Theorem 5.2 and Lemma 3.10 (of course, $T=T_{0,\mathscr{A}}$).

Theorem 5.3 ensures the possibility of a first-order infinitary axiomatization of some second-order concepts of a topological kind.

5.4. Example. Consider the concrete category Top of topological spaces and continuous maps. It is well known that ultraspaces form a finally dense class in Top. Recall that ultraspaces are topological spaces $(\mathscr{G},x)$ having $\mathscr{G}\cup 2^{X-\{x\}}$ as the system of open sets where X is the underlying set, $x\in X$ and $\mathscr{G}$ an ultrafilter on X distinct from the principal ultrafilter $X(x)$ generated by x.

Denote by $\mathscr{C}_n$ the set of all non-isomorphic ultraspaces on n of the kind $(\mathscr{G},0)$. Form the relational type τ by taking $\mathrm{Rel}_n(\tau)=\{R_{\mathscr{G}}\mid(\mathscr{G},0)\in\mathscr{C}_n\}$. Following Theorem 5.3, there is a universal strict Horn theory T of type τ without equality such that $\mathrm{Top}\cong\mathrm{Mod}(T)$. A nice explicit presentation of T is included in Manes [21]. If we rewrite its description in our terms, we get that T may be given by sentences (in what follows, if $x\in V^n$ and $z\in V$, then x^z denotes the n-tuple $(z,x_1,x_2,\dots,x_i,\dots)$, $i\in n$):

$$(\forall x)(R_{f(\mathscr{G})}(x)\Rightarrow R_{\mathscr{G}}(x\cdot f)) \tag{5.2}$$

for any $f\in m^n$ and any $(\mathscr{G},0)\in\mathscr{C}_n$ ($f(\mathscr{G})$ is the ultrafilter on m generated by $\{f(X)\mid X\in\mathscr{G}\}$),

$$(\forall x)(R_{n(i)}(x^{x_i}) \tag{5.3}$$

for any $0\neq i\in n$, where $x\in V^n$,

$$(\forall x,y)\left(\left(\bigwedge_{0\neq i\in n}R_{\mathscr{G}_i}(x^{y_i})\right)\wedge R_{\mathscr{G}}(y)\Rightarrow R_{\mathscr{F}}(x^y)\right) \tag{5.4}$$

where $x \in V^m$, $y \in V^n$, $(\mathscr{G}_i, 0) \in \mathscr{C}_m$ for $0 \neq i \in n$, $(\mathscr{G}, 0) \in \mathscr{C}_n$ and $(\mathscr{F}, 0) \in \mathscr{C}_m$ such that $X \in \mathscr{F}$ iff there is $K \in \mathscr{G}$ such that $X \in \mathscr{G}_i$ for any $i \in K$.

Other examples can be extracted from Dubuc [10] and [11] because his quasi-spaces over a concrete site $\mathscr{A}$ are precisely models of the following theory of the type $\tau_{\mathscr{A}_*}$:

$$(\forall x) R_1(x)$$

where 1 is the terminal object of $\mathscr{A}$,

$$(\forall x)(R_D(x) \Rightarrow R_C(x \cdot f))$$

where $f: C \to D$ and $C, D \in \mathscr{A}_*$,

$$(\forall x)\left(\bigwedge_{i \in I} R_{C_i}(x \cdot f_i) \Rightarrow R_C(x)\right)$$

where $(f_i: C_i \to C)_{i \in I}$ is a covering of a Grothendieck topology on $\mathscr{A}$.

A *monosource* is a source $(f_i: X \to |A_i|)$ such that for any $u, v \in X$, $u \neq v$ there is an i with $f_i(u) \neq f_i(v)$.

A concrete category will be called *weakly initially complete* if each monosource has an initial lift. Weakly initially complete categories were introduced by Herrlich [12] (under the name (epi, monosource)-topological categories).

5.5. Proposition. *Let T be a relational universal strict Horn theory. Then* Mod(T) *is weakly initially complete.*

Proof. Consider a monosource $(f_i: A \to |\mathfrak{A}_i|)$ and follow the proof of Proposition 5.1. New sentences which T may contain in addition to (1) and (2) are of the kinds

$$(\forall x)\left(\bigwedge_{j \in J} R_j(x \cdot g_j) \Rightarrow x_r = x_s\right)$$

where the left side is the same as in (2) and $r, s \in J$,

$$(\forall x)(\forall y)(x = y).$$

However, since we have started from a monosource, these sentences hold in $\mathfrak{A}$.

A *weak initial completion* of $\mathscr{A}$ is a weakly initially complete category containing $\mathscr{A}$ as a full subcategory.

If $\mathscr{C} \subseteq \mathscr{A}_*$ then $T_{1,\mathscr{C}}$ will denote the theory of type $\tau_{\mathscr{C}}$ consisting of all strict Horn sentences from $T_{\mathscr{C}}$.

5.6. Theorem. *Let $\mathscr{A}$ be strongly fibre-small and $\mathscr{C} \subseteq \mathscr{A}_*$ be finally dense. Then* Mod$(T_{1,\mathscr{C}})$ *is the smallest weak initial completion of $\mathscr{A}$.*

Proof. Mod$(T_{1,\mathscr{C}})$ is a weak initial completion of $\mathscr{A}$ by Proposition 5.5 and 3.3, Lemma 3.10 and Corollary 2.6. We show that any $T_{1,\mathscr{A}}$-model $\mathfrak{B}$ is an initial lift of a monosource with codomains in $\mathscr{A}$. Since we know from Theorem 5.2 that $\mathscr{A}$ is initially dense in $\mathrm{Mod}(T_{1,\mathscr{A}})$, it suffices to verify that for any two distinct elements $i, j \in |\mathfrak{B}|$ there is $A \in \mathscr{A}$ and a homomorphism $f: \mathfrak{B} \to A$ such that $f(i) \neq f(j)$. Suppose $f(i) = f(j)$ for all $f: \mathfrak{B} \to A$, $A \in \mathscr{A}$. Then the sentence

$$(\forall x)\left[\left(\bigwedge_{(C,p)\in P(\mathfrak{B})} R_C(x \cdot p)\right) \Rightarrow x_i = x_j\right]$$

belongs to $T_{1,\mathscr{A}}$ by (5.1). Therefore $i = j$ because $\mathfrak{B} \models \bigwedge_{(C,p)\in P(\mathfrak{B})} R_C[p]$.

Let $\mathscr{B}$ be a weak initial completion of $\mathscr{A}$. Consider a $T_{1,\mathscr{A}}$-model $\mathfrak{A}$. Let $H(\mathfrak{A})$ be an initial lift in $\mathscr{B}$ of the monosource $\{f: |\mathfrak{A}| \to |A|\}$ consisting of all homomorphisms $f: \mathfrak{A} \to A$ where $A \in \mathscr{A}$. By initiality, H can be extended to a functor $\mathrm{Mod}(T_{1,\mathscr{A}}) \to \mathscr{B}$. The restriction of H on $\mathscr{A}$ is evidently the identity.

Assume that $g: H(\mathfrak{A}) \to A$ is a homomorphism. Then $g \cdot h$ is a homomorphism $B \to A$ for any homomorphism $h: B \to \mathfrak{A}$. Hence $g: \mathfrak{A} \to A$ is a homomorphism because G is finally dense by Theorem 5.2. Since $G_{\mathscr{A}}$ is initially dense, H has to be full. Hence $\mathrm{Mod}(T_{1,\mathscr{A}})$ is the smallest weak initial completion of $\mathscr{A}$.

Let us remark that $\mathscr{B}$ is a weak initial completion of $\mathscr{A}$ iff $\mathscr{B}$ is a weakly initially complete category containing $\mathscr{A}$ as a full finally dense subcategory such that any object of $\mathscr{B}$ is an initial lift of a monosource with codomains in $\mathscr{A}$. Further, Theorem 5.2 yields that $\mathscr{A}$ has a fibre-small weak initial completion iff it is strongly fibre-small.

The next theorem was stated in [27].

5.7. Theorem. *The following two conditions are equivalent for any concrete category* $\mathscr{A}$:

(i) $\mathscr{A}$ *is a fibre-small weakly initially complete category.*

(ii) $\mathscr{A} \cong \mathrm{Mod}(T)$ *for a relational, normal, universal strict Horn theory* T.

Proof. Analogous to that of Theorem 5.3.

6. Semi-initially complete categories

Let $\mathscr{A}$ be a concrete category. A *semi-final lift* of a sink $(f_i: |A_i| \to X)_{i\in I}$ consists of an object $A \in \mathscr{A}$ and of a mapping $e: X \to |A|$ such that $e \cdot f_i$ is a morphism $A_i \to A$ for any $i \in I$ and these data have the following universal property: for any $B \in \mathscr{A}$ and $g: X \to |B|$ such that $g \cdot f_i$ is a morphism $A_i \to B$ for any $i \in I$ there is a unique morphism $t: A \to B$ with $t \cdot e = g$. The mapping e is called the *quotient* of a sink (f_i). $\mathscr{A}$ is called *semi-initially complete* if any sink $(f_i: |A_i| \to X)$ has a semi-final lift. Semi-initially complete categories were independently introduced under different names by R.-E. Hoffmann, Tholen, Trnková and Wischnewsky. We keep the initial terminology though the now prevailing name is 'semi-topological' and though we have defined them by semi-final lifts. However, the definition by an appropriate concept of a semi-initial lift is also possible (see Tholen [32]).

It is known (see [32, 6.8]) that a co-well-powered category is semi-initially complete iff it is cocomplete and has free objects. It can be shown that any co-well-powered semi-initially complete category is strongly fibre-small. On the other hand, there is a strongly fibre-small semi-initially complete category which is not co-well-powered. Namely, following Isbell [14, 3.12], local lattices form a non-co-well-powered varietal category. The just quoted characterization of semi-initiality by means of cocompleteness and free objects can be immediately extended to strongly fibre-small categories.

6.1. Lemma. *Any strongly fibre-small cocomplete concrete category having free objects is semi-initially complete.*

Proof. Since $\mathscr{A}$ is strongly fibre-small, it suffices to find semi-final lifts to sinks which are sets. So let $(f_i : |A_i| \to X)_{i\in I}$ be a sink and I a set. Denote by F a left adjoint to $|\ \ |$, by ε its co-unit and by $\varphi : \mathscr{A}(FX, A) \to |A|^X$ the adjunction isomorphism. Denote by $v_i : A_i \to \sum_{i\in I} A_i$, $u_i : F(|A_i|) \to \sum_{i\in I} F(|A_i|)$ the injections and by $f : \sum_{i\in I} F(|A_i|) \to F(X)$ the unique morphism such that $f \cdot u_i = F(f_i)$ for all $i \in I$. It is easy to show that

$$\begin{array}{ccc} \sum F|A_i| & \xrightarrow{f} & FX \\ {\scriptstyle \sum \varepsilon_{A_i}} \downarrow & & \downarrow {\scriptstyle e} \\ \sum A_i & \xrightarrow{\bar{f}} & A \end{array}$$

is a push-out iff

$$\begin{array}{ccc} |A_i| & \xrightarrow{f_i} & X \\ & {\scriptstyle |\bar{f}\cdot v_i|} \searrow \quad \swarrow {\scriptstyle \varphi(e)} & \\ & |A| & \end{array}$$

yields a semi-final lift of (f_i).

6.2. Proposition. *Let T be a normal strongly tight strict $\exists!$-theory. Then* $\mathrm{Mod}(T)$ *is semi-initially complete.*

Proof. Although it would be possible to give a direct proof, we are going to use Lemma 6.1 in order to prepare the proof of Theorem 6.5.

Let I be a set and $\mathfrak{A}_i$ T-models for any $i \in I$. Let $\bar{\tau}$ be the type from Proposition 4.7 where τ denotes the type of T. The disjoint union $\mathfrak{A}$ of $\mathfrak{A}_i$ (i.e. with the structure of $\mathfrak{A}_i$ on each summand) is a $\bar{\tau}$-model such that homomorphisms $\mathfrak{A} \to U(\mathfrak{B})$ of $\bar{\tau}$-models correspond to families $(\mathfrak{A}_i \to \mathfrak{B})_{i\in I}$ of homomorphisms of τ-models for any $\mathfrak{B}\in \mathrm{Mod}(\tau)$. Hence the sum $\sum_{i\in I}\mathfrak{A}_i$ is equal to $F(\mathfrak{A})$, which exists by Proposition 4.7. The canonical injections $\mathfrak{A}_i \to F(\mathfrak{A})$ are given by the unit $\mathfrak{A}\to UF(\mathfrak{A})$ of the adjunction.

Consider homomorphisms $u, v: \mathfrak{B}\to\mathfrak{C}$ of T-models and let $e: |\mathfrak{C}| \to A$ be a coequalizer in Set of

$$|\mathfrak{B}| \overset{u}{\underset{v}{\rightrightarrows}} |\mathfrak{C}|.$$

Let $\mathfrak{A}$ be a $\bar{\tau}$-model on X which is the image of $\mathfrak{C}$ in e (i.e. $R_{\mathfrak{A}} = \{e\cdot c \mid c\in R_{\mathfrak{C}}\}$ for any $R\in \mathrm{Rel}(\tau)$ and $\mathfrak{A}\models R_f[a,b]$ iff there is $c\in|\mathfrak{C}|^n$ such that $b=e\cdot c$ and $a=e(f_{\mathfrak{C}}[c])$). Then $F(\mathfrak{A})$ gives the desired coequalizer of u, v.

We have proved that $\mathscr{A}$ is cocomplete. Since it has free objects by Corollary 4.9 and is strongly fibre-small following Corollary 2.6, Lemma 6.1 completes the proof.

A *semi-initial completion* of $\mathscr{A}$ is a semi-initially complete category containing $\mathscr{A}$ as a full subcategory.

If $\mathscr{C} \subseteq \mathscr{A}_*$, then $T_{4,\mathscr{C}}$ will denote the theory of type $\tau_{\mathscr{C}}$ consisting of all sentences (4.6)

$$(\forall x)(\exists y)(\varphi(x) \Rightarrow \psi(x,y))$$

such that for any $A\in\mathscr{A}$

$$A \models (\forall x)[\varphi(x) \Rightarrow (\exists! y)\psi(x,y)].$$

6.3. Theorem. *Let $\mathscr{A}$ be strongly fibre-small and $\mathscr{C}\subseteq\mathscr{A}_*$ finally dense. Then* $\mathrm{Mod}(T_{4,\mathscr{C}})$ *is the smallest semi-initial completion of $\mathscr{A}$ in which $\mathscr{A}$ is finally dense.*

Proof. Denote $T_{4,\mathscr{C}}$ briefly by T. We begin with the proof that T is strongly tight. Let $\varrho(x)$ be a diagrammatic formula of type $\tau_{\mathscr{C}}$ where $x\in V^n$. Then $\varrho(x)$ is of the kind $\bigwedge_{i\in I} R_{C_i}(x\cdot f_i)$ where $C_i\in\mathscr{C}$ and $f_i \in n^{C_i}$ for $i\in I$. Since $\mathscr{A}$ is strongly fibre-small and thus strongly co-fibre-small as well, the source consisting of $g: n\to |A|$ such that $A\in\mathscr{A}$ and $A\models\varrho[g]$ has a $\sim$-representative set S. Consider two $\exists\bigwedge$-formulas $\varphi(x,y)$ and $\psi(x,y)$ satisfying (4.12) and such that

$$A\models(\forall y)(\varphi(g,y)\leftrightarrow\psi(g,y)) \tag{6.1}$$

for any $(g,A)\in S$. Then the formulas $\varphi(x,y)$ and $\psi(x,y)$ have to be ϱ-equivalent. Indeed, whenever $B\in\mathscr{A}$ and $b\in|B|^n$ with $B\models\varrho[b]$, then there is $(g,A)\in S$ equivalent to (b,B). It follows however that $A\models\alpha[g]$ iff $B\models\alpha[b]$ for any formula $\alpha(x)$ with $x\in V^n$ ($(g,A)\sim(b,B)$ immediately gives that it holds for atomic formulas). Hence $B\models(\forall y)(\varphi(b,y)\Leftrightarrow\psi(b,y))$, which proves the ϱ-equivalence of φ and ψ. Since there is only a set of formulas for which (6.1) does not hold, T is strongly tight.

To show that T is a strict $\exists!$-theory we need to know that

$$\mathfrak{A}\models(\forall x)[\varphi(x)\Rightarrow(\exists! y)\psi(x,y)]$$

holds for any sentence from T and any T-model $\mathfrak{A}$. But this is true because it holds for T-models $A \in \mathscr{A}$ and any T-model is, following Theorem 5.6, an initial lift of a monosource with codomains in $\mathscr{A}$. Hence using Propositions 6.2 and 3.3, Lemma 3.10, Corollary 2.6 and Theorem 5.2, we get that $\mathrm{Mod}(T)$ is a semi-initial completion of $\mathscr{A}$ in which $\mathscr{A}$ is finally dense.

Let $\mathscr{B}$ be a semi-initial completion of $\mathscr{A}$ in which $\mathscr{A}$ is finally dense. Consider a T-model $\mathfrak{A}$. Let $H(\mathfrak{A})$ and $e: |\mathfrak{A}| \to |H(\mathfrak{A})|$ be a semi-final lift in $\mathscr{B}$ of the sink $P(\mathfrak{A})$. If we prove that e is a bijection then we will see that $H: \mathrm{Mod}(T) \to \mathscr{B}$ is a functor. Thus the proof will be accomplished because the fullness of H is given analogously to Theorem 5.6.

Let Q be a $\sim$-representative set of the sink consisting of all morphisms $g: C \to H(\mathfrak{A})$ in $\mathscr{B}$ with $C \in \mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{A}$ and $\mathscr{A} \subseteq \mathscr{B}$ are finally dense, $\mathscr{C} \subseteq \mathscr{B}$ is finally dense, too. Hence $b: H(\mathfrak{A}) \to A$ is a morphism iff $b \cdot g: C \to A$ is a morphism for any $(C, g) \in Q$. Therefore (5.1) and the semifinality imply that the sentence

$$(\forall x)(\exists! y)\left(\bigwedge_{(C,p) \in \bar{P}(\mathfrak{A})} R_C(x \cdot p) \Rightarrow \bigwedge_{(C,g) \in Q} R_C(y \cdot g) \wedge y \cdot e = x\right) \tag{6.2}$$

belongs to T.

Since $\mathfrak{A} \models \bigwedge_{(C,p) \in \bar{P}(\mathfrak{A})} R_C[p]$, (6.2) provides a unique $a: |H(\mathfrak{A})| \to |\mathfrak{A}|$ such that $a \cdot e = 1_{|\mathfrak{A}|}$ and $\mathfrak{A} \models \bigwedge_{(C,g) \in Q} R_C[a \cdot g]$. Hence $a \cdot g \in P(\mathfrak{A})$ for any $(C, g) \in Q$ and thus $e \cdot a \cdot g: C \to H(\mathfrak{A})$ is a homomorphism for any $(C, g) \in Q$. Since $\mathscr{C}$ is finally dense in $\mathscr{B}$, $e \cdot a: H(\mathfrak{A}) \to H(\mathfrak{A})$ is a morphism in $\mathscr{B}$. Since $(e \cdot a) \cdot e = e = 1_{H(\mathfrak{A})} \cdot e$, e being a quotient yields that $e \cdot a = 1_{H(\mathfrak{A})}$. Hence a is the inverse to e and the proof is accomplished.

It is easy to see that $\mathscr{A}$ is finally dense in its semi-initial completion $\mathscr{B}$ iff morphisms $f: A \to B$, $A \in \mathscr{A}$, are jointly onto for any $B \in \mathscr{B}$. We may hence assign to any semi-initial completion $\mathscr{B}$ of $\mathscr{A}$ a semi-initial completion of $\mathscr{A}$ in which $\mathscr{A}$ is finally dense, by restricting the underlying sets of objects of $\mathscr{B}$ to unions of images of morphisms going from $\mathscr{A}$.

6.4. Theorem. *The following conditions are equivalent for any concrete category* $\mathscr{A}$:

(i) $\mathscr{A}$ *is cocomplete, strongly fibre-small and has free objects,*
(ii) $\mathscr{A}$ *is strongly fibre-small and semi-initially complete,*
(iii) $\mathscr{A} \cong \mathrm{Mod}(T)$ *for a normal, strongly tight strict* $\exists!$*-theory* T.

Proof. (i)$\Rightarrow$(ii) is Lemma 6.1, (ii)$\Rightarrow$(iii) follows from Theorem 6.3, and (iii)$\Rightarrow$(i) is given by the proof of Proposition 6.2.

The next theorem was stated in [26] and [27] (with strong fibre-smallness incorrectly replaced by $\mathscr{A}$ being co-well-powered). The proof in [26] follows the reasoning of Keane [15] and goes via Freyd's adjoint functor theorem. The following proof is more natural and may be carried out over other base categories than Set because in verifying (ii)$\Rightarrow$(iii) we do not need cocompleteness.

6.5. Theorem. *The following conditions are equivalent for any concrete category* $\mathscr{A}$:

(i) $\mathscr{A}$ *is cocomplete, strongly fibre-small, has free objects and satisfies:*

(a) *If* $\bar{f}\cdot g=\bar{g}\cdot f$ *is a push-out in* $\mathscr{A}$ *and* $|f|$ *is onto, then* $|\bar{f}|$ *is onto* (*i.e. push-outs preserve onto morphisms*).

(b) *If* I *is a set and* $|f_i|:|A_i|\to|B_i|$ *are onto for any* $i\in I$, *then*

$$\left|\sum_{i\in I} f_i\right|:\left|\sum_{i\in I} A_i\right|\to\left|\sum_{i\in I} B_i\right|$$

is onto.

(ii) $\mathscr{A}$ *is strongly fibre-small, semi-initially complete and quotients of sinks containing an onto mapping are onto.*

(iii) $\mathscr{A}\cong\mathrm{Mod}(T)$ *for a normal universal strict Horn theory* T.

Proof. Assume (i) and follow the proof of Lemma 6.1. Let (f_i) contain an onto mapping f_{i_0}. Using the preservation of onto mappings by set functors, assumptions (a), (b) and properties of an adjunction, we successively obtain that the following mappings are onto: $|F(f_{i_0})|$, $|\varepsilon_{A_i}|$, $|\sum\varepsilon_{A_i}|$, e, $|\bar{f}\cdot v_{i_0}\cdot\varepsilon_{A_{i_0}}|=|\bar{f}\cdot\sum\varepsilon_{A_i}\cdot u_{i_0}|=e\cdot|f\cdot u_{i_0}|=e\cdot|F(f_{i_0})|$, $\varphi(e)\cdot f_{i_0}=|\bar{f}\cdot v_{i_0}|$ and $\varphi(e)$. Therefore (ii) holds.

Assume (ii). Let $\mathscr{C}\subseteq\mathscr{A}_*$ be finally dense and denote by $T=T_{3,\mathscr{C}}$ the theory of type $\tau_{\mathscr{C}}$ consisting of all universal strict Horn sentences which hold in $\mathscr{A}$ and of sentences $(\forall x)(\exists y)\varphi(x,y)$ where φ is a $\wedge$-sentence such that for any $A\in\mathscr{A}$

$$A\models(\forall x)(\exists!y)\varphi(x,y).$$

(iii) will be verified whenever we show that $\mathscr{A}\cong\mathrm{Mod}(T)$. Indeed, form an expansion σ of τ by assigning to each $\wedge$-formula $\varphi(x,y)$ such that $x\in V^n$, $y\in V$ and $A\models(\forall x)(\exists!y)\varphi(x,y)$ a new n-ary function symbol $\bar{\varphi}$. Consider the coherent extension S of type σ of T given by sentences $(\forall x)(\forall y)(y=\bar{\varphi}(x)\Leftrightarrow\varphi(x,y))$. It is clear that S is a universal strict Horn theory. With regard to Lemma 4.1 it suffices to check that S is normal. Since S and T have the same equivalence classes of quantifier-free formulas and T is normal by Lemma 3.10, S has to be normal.

Consider a T-model $\mathfrak{A}$. Let A, $e:|\mathfrak{A}|\to|A|$ be a semi-final lift of a sink $P(\mathfrak{A})$. By semi-finality, any homomorphism $\mathfrak{A}\to X$, $X\in\mathscr{A}$, can be factorized through e. Since homomorphisms $\mathfrak{A}\to X$, $X\in\mathscr{A}$, form a monosource by the proof of Theorem 5.6, e has to be mono. Further, $e:\mathfrak{A}\to A$ is a homomorphism following (5.1).

Let $D\in\mathscr{C}$ and $a\in|\mathfrak{A}|^{|D|}$ such that $e\cdot a\in(R_D)_A$. Then

$$T\models(\forall x)\left(\bigwedge_{(C,p)\in P(\mathfrak{A})} R_C(x\cdot p)\Rightarrow R_D(x\cdot a)\right).$$

Indeed, if $B\in\mathscr{A}$ and $b\in|B|^{|\mathfrak{A}|}$ such that $B\models\bigwedge_{(C,p)\in P(\mathfrak{A})}R_D[b\cdot p]$ then $b:\mathfrak{A}\to B$ is a homomorphism by (5.1) and thus there is a homomorphism $h: A \to B$ such that $h \cdot e = b$. Hence $b \cdot a = h \cdot e \cdot a \in (R_D)_B$. Since $\mathfrak{A} \models \bigwedge_{(C,p) \in \bar{P}(\mathfrak{A})} R_C[p]$, we get that $\mathfrak{A} \models R_D[a]$.

We have proved that $e: \mathfrak{A} \to A$ is an isomorphic embedding. It remains to show that e is onto, i.e. that $P(\mathfrak{A})$ contains an onto mapping. Denote by F a left adjoint to the underlying functor of $\mathscr{A}$ and by η the unit of the adjunction. Hence for any $B \in \mathscr{A}$ and any $b: |\mathfrak{A}| \to |B|$ there is a unique homomorphism $d: F(|\mathfrak{A}|) \to B$ such that $d \cdot \eta_{|\mathfrak{A}|} = b$. Syntactically, it means that for any $B \in \mathscr{A}$

$$B \models (\forall x)(\exists ! y)\Big(\bigwedge_{(C,p) \in \bar{P}(F(\mathfrak{A}))} R_C(y \cdot p) \wedge x = y \cdot \eta_{|\mathfrak{A}|}\Big).$$

Hence the sentence

$$(\forall x)(\exists y)\Big(\bigwedge_{(C,p) \in \bar{P}(F(\mathfrak{A}))} R_C(y \cdot p) \wedge x = y \cdot \eta_{|\mathfrak{A}|}\Big)$$

belongs to T. By substituting $1_{|\mathfrak{A}|}$ for x we get that there is a homomorphism $g: F(|\mathfrak{A}|) \to \mathfrak{A}$ such that $g \cdot \eta_{|\mathfrak{A}|} = 1_{|\mathfrak{A}|}$. Therefore g is a desired onto mapping from $P(\mathfrak{A})$. Hence (iii) holds.

Assume (iii). Then Mod(T) is strongly fibre-small by Corollary 2.6.

Let S be an arbitrary normal universal strict Horn theory of type σ. We show that Mod(S) has free objects. Let n be a cardinal and A be an S-representative set of atomic formulas $y = t(x)$ where t is a term and $x \in V^n$. Elements of A will be identified with the corresponding terms t. It is easy to show that the following σ-model $\mathfrak{A}$ on A is the desired free object over $n: f_{\mathfrak{A}}(t_1, \ldots, t_i, \ldots) \sim f(t_1, \ldots, t_i, \ldots)$ for any $k \in \mathrm{Card}$, $f \in \mathrm{Fnt}_k(\sigma)$, $t_i \in A$ and $i \in k$; $\mathfrak{A} \models R[t_1, \ldots, t_i, \ldots]$ iff $S \models R(t_1(x), \ldots, t_i(x), \ldots)$ for any $k \in \mathrm{Card}$, $R \in \mathrm{Rel}_k(\sigma)$, $t_i \in A$ and $i \in k$.

In particular, Mod(T) has free objects. Since $T(\varrho(c))$ is a normal universal strict Horn theory for any diagrammatic formula $\varrho(x)$, $x \in V^n$, and constants c_i, $i \in n$, Mod$(T(\varrho(c))$ has free objects, as well. Hence T is strongly tight by Lemmas 4.6 and 4.4 because the free object over the empty set is an initial object in Mod(T). Therefore Mod(T) is cocomplete by Theorem 6.4. It remains to verify (a) and (b).

Following the proof of Proposition 6.2, push-outs in Mod(T) are constructed as follows. Let $f: \mathfrak{B} \to \mathfrak{C}$ and $g: \mathfrak{B} \to \mathfrak{D}$ be homomorphisms of T-models and f be onto. Let $f': \mathfrak{D} \to \mathfrak{A}$, $g: \mathfrak{C} \to \mathfrak{A}$ give a push-out of $U(f)$, $U(g)$ in Mod$(\bar{\tau})$. Let $\eta_{\mathfrak{A}}: \mathfrak{A} \to UF(\mathfrak{A})$ be the unit of the adjunction from Proposition 4.7. Since U is full, there are unique morphisms $\bar{f}: \mathfrak{D} \to F(\mathfrak{A})$ and $\bar{g}: \mathfrak{C} \to F(\mathfrak{A})$ such that $U(\bar{f}) = \eta_{\mathfrak{A}} \cdot f'$ and $U(\bar{g}) = \eta_{\mathfrak{A}} \cdot g'$. Then $\bar{f}$, $\bar{g}$ give the push-out of f, g in Mod(T). Since $|\ |: \mathrm{Mod}(\bar{\tau}) \to \mathrm{Set}$ creates colimits, f' is onto. Following the construction of $F(\mathfrak{A})$, $|F(\mathfrak{A})|$ consists of all 0-ary terms of the theory $T(\varrho(c))$ where $\varrho(c)$ has the meaning from the proof of Proposition 4.7. These terms are inductively constructed from the constants of type τ and from c_a, $a \in |\mathfrak{A}|$, using function symbols of τ. But since f' is onto, it implies that $\eta_{\mathfrak{A}} \cdot f'$ is onto, as well. Hence (a) holds.

Let $u_i: \mathfrak{A}_i \to \mathfrak{B}_i$ be onto homomorphisms of T-models for any $i \in I$. Since $|\sum \mathfrak{A}_i|$ and

$|\sum \mathfrak{B}_i|$ consist of all 0-ary terms of theories given by the proof of Propositions 6.2 and 4.7, $\sum u_i$ has to be onto.

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